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INSTITUTT FOR ENERGI- OG PROSESSTEKNIKK

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EXAM TEP4145 KLASSISK MEKANIKK
Monday 21. May 2007 kl. 0900 - 1300
English version

Remedies: C

- K. Rottmann: Mathematical formulae
- Approved calculator, with empty memory, according to list worked out by NTNU. (HP30S or similar.)

Appendix A: The questions (Page 2 - 6).
Appendix B: Formulas (Page 7 - 8).

The exam consists of 5 questions. The weight for each question is given. Do questions 1, 2 and 3, and either question 4 (physics) or 5 (cybernetics).
Note that Einstein's summation convention is used unless otherwise stated, i.e., summation over repeated indices.

The grades will be ready around June 12.

Appendix A: The questions

QUESTION 1 [Counts 25%]

a) Derive Lagrange's equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0,$$

from Hamilton's principle

$$\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0.$$

Here, there is no variation in the end points, and virtual variations in the coordinate q is performed at fixed time t, i.e., $\delta t = 0$.

b) A point mass m slides without friction on a ring which rotates in the xy plane with constant angular velocity ω_0. The ring, with radius R, rotates around the origin $x = y = 0$, as shown in the figure:

• Find the lagrangian $L(\theta, \dot{\theta})$ of the mass m.
• Derive Lagrange's equation, i.e., the equation of motion for the mass m.
• The equation of motion would have been the same if the mass m were swinging back and forth at the end of a massless rod in the field of gravity (acceleration of gravity $= g$). Verify this, and determine the length l of the rod.

QUESTION 2 [Counts 25%]

Three spheres with mass m (sphere nr 1, to the left), $3m$ (sphere nr 2, in the middle), and $2m$ (sphere nr 3, to the right), respectively, are connected via two identical (and ideal) springs with spring constant k, as shown in the figure:

The springs can only be stretched, not bent. We assume that the spheres may only move along the x axis, and we will here investigate oscillations around the equilibrium positions of the spheres. These are x_{10}, x_{20}, and x_{30}, respectively.

• Use the deviations from equilibrium, $\eta_i = x_i - x_{i0}$ $(i = 1, 2, 3)$, as coordinates, and find the (symmetric) matrices $\boldsymbol{V}$ and $\boldsymbol{T}$ in the quadratic expressions

$$\begin{aligned} V &= \frac{1}{2} V_{ij}\, \eta_i\, \eta_j \\ T &= \frac{1}{2} T_{ij}\, \dot{\eta}_i\, \dot{\eta}_j \end{aligned}$$

for the potential and the kinetic energy of the system. Here, V_{ij} and T_{ij} are matrix elements of $\boldsymbol{V}$ and $\boldsymbol{T}$, respectively.

• Solve the secular equation,

$$\left| \boldsymbol{V} - \omega^2\, \boldsymbol{T} \right| = 0,$$

and thus determine the two eigenfrequencies of the system, $f_\alpha = \omega_\alpha / 2\pi$ $(\alpha = 1, 2)$. Find numerical values of f_α when $m = 100$ g and $k = 10^3$ N/m. (Neglect the mode with $\omega = 0$, which corresponds to pure translation of the system.)

QUESTION 3 [Counts 25%.]

In this question, we will study a canonical transformation $(q, p) \to (Q, P)$ of a simple one dimensional harmonic oscillator ($i = \sqrt{-1}$):

$$
\begin{aligned}
Q(q,p) &= \frac{1}{\sqrt{2i}}(q + ip) \\
P(q,p) &= \frac{-1}{\sqrt{2i}}(q - ip)
\end{aligned}
$$

For simplicity, we use $m = k = 1$ (m = mass and k = spring constant of the oscillator).

- What is the hamiltonian $H(q, p) = T + V$ of this oscillator?
- Show that the given transformation $(q, p) \to (Q, P)$ *is* canonical. Hint: Use the fact that Poisson brackets are invariant under a canonical transformation, for example $[q, p] = [Q, P]$.
- Find the hamiltonian $K(Q, P)$ in the new coordinates Q, P. (Here, $K = H$)
- Find Hamilton's equations for Q and P, and solve these equations, using the initial conditions $q(t = 0) = p(t = 0) = 1$.
- What is the total energy of the oscillator?

QUESTION 4 [For the physics students, counts 25%.]

a) The electromagnetic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(see formulas in appendix B for A_μ, and its relation to $\boldsymbol{E}$ and $\boldsymbol{B}$). Find the elements of the matrix $\boldsymbol{F}$ (i.e.: expressed in terms of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$).

b) Show that $\boldsymbol{E} \cdot \boldsymbol{B}$ is invariant under a Lorentz transformation. (Use the transformation equations for the electromagnetic field given in appendix B.)

c) A point charge q passes through the origin at time $t = 0$ and moves with constant velocity v in the positive x direction. Let us with S_0 denote the inertial system where the point charge is at rest, and with S denote the inertial system where the point charge moves with velocity $v\hat{x}$. The electric field at a distance $\boldsymbol{r}_0$ from the point charge is obviously

$$\boldsymbol{E}_0 = \frac{q}{4\pi\varepsilon_0} \frac{\boldsymbol{r}_0}{r_0^3}$$

measured in S_0. Furthermore, the electric field at a distance $\boldsymbol{r}$ from the point charge is

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2\theta)^{3/2}} \frac{\boldsymbol{r}}{r^3}$$

measured in S. Here, θ denotes the angle between the x axis and $\boldsymbol{r}$, see figure below. Show that the magnetic field, measured in S, may be written as

$$\boldsymbol{B} = \frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{E}$$

d) Show that the magnetic field $\boldsymbol{B}$ from the moving point charge reduces to

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} \frac{q\boldsymbol{v} \times \boldsymbol{r}}{r^3}$$

in the nonrelativistic limit $v \ll c$. Sketch field lines for $\boldsymbol{B}$ in the yz plane.

QUESTION 5 [For cybernetics students, counts 25%.]

a) Let us investigate how the stock of a certain animal will develop in time. Assume - in the simplest possible manner - that this number increases at a steady rate each year, i.e., there exists a $\mu > 1$ such that

$$b_{n+1} = \mu \, b_n,$$

where b_n is the stock in year n and b_{n+1} is the stock in year $n+1$. What kind of equation is this, and what is its general solution? Explain briefly why this model is unrealistic.

b) An improved model is obtained by letting B be an upper limit of the number of animals, and defining the model by

$$b_{n+1} = \mu \, b_n \left(1 - \frac{b_n}{B}\right).$$

We introduce the relative stock,

$$x_n = \frac{b_n}{B},$$

and hence we have

$$x_{n+1} = \mu \, x_n \left(1 - x_n\right).$$

Next, we define

$$F_\mu(x) = \mu \, x \left(1 - x\right),$$

which yields

$$x_{n+1} = F_\mu(x_n).$$

Make a simple sketch of $F_\mu(x)$, and write down the coordinates of the maximum point of the curve. Let x_0 be the intersection between the graph $y = F_\mu(x)$ and the graph $y = x$. Then, what is x_n as a function of x_0? What kind of point is x_0?

c) Define the concept "fixed point" for a function f. Define the concept "periodic point" for a function f, and explain what is meant by "grunnperiode" for the point. Explain/define "attractive fixed point" and "repulsive fixed point" for a function f.

d) Finally, consider the function

$$F_\mu(x) = \mu \, x \left(1 - x\right).$$

Determine the two fixed points of F_μ. Are the fixed points attractive or repulsive? Summarize your answer by providing restrictions on μ. Let the upper limit of μ be 3. (Proofs are not necessary.)

Appendix B: Formulas

The validity of the formulas and the meaning of the symbols are assumed to be known.

- Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

- Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

- Poisson brackets:

$$[f, g] = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$$

$$[q_i, q_j] = [p_i, p_j] = 0$$

$$[q_i, p_j] = \delta_{ij}$$

- Four-vector:

$$x_\mu = (\boldsymbol{r}, ict)$$

- Four-potential:

$$A_\mu = (\boldsymbol{A}, i\phi/c)$$

- Electromagnetic field:

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

- Lorentz transformation (relative velocity $\boldsymbol{v} = v\hat{x}$):

$$L_{22} = L_{33} = 1 \;\; , \;\; L_{11} = L_{44} = \gamma \;\; , \;\; L_{14} = -L_{41} = i\beta\gamma$$

$$\beta = v/c \quad , \quad \gamma = 1/\sqrt{1 - v^2/c^2}$$

- Lorentz transformation of electromagnetic field (inertial system S_0 moves with velocity $v\hat{x}$ relative to S):

$$\begin{aligned} E_x &= E_{0x} \\ E_y &= \gamma(E_{0y} + vB_{0z}) \\ E_z &= \gamma(E_{0z} - vB_{0y}) \\ B_x &= B_{0x} \\ B_y &= \gamma(B_{0y} - \frac{v}{c^2}E_{0z}) \\ B_z &= \gamma(B_{0z} + \frac{v}{c^2}E_{0y}) \end{aligned}$$

- Trigonometric relations:

$$\begin{aligned} \cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\ \sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \end{aligned}$$