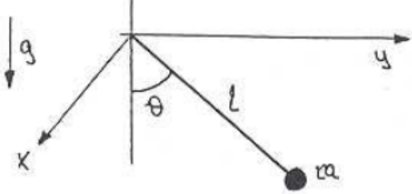



2) $\theta$ er supplementsvinkelen til ordinær polarvinkel

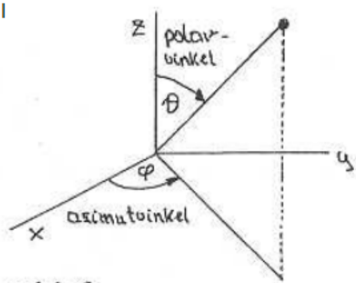
$x = l\sin\theta\cos\varphi$

$y = l\sin\theta\sin\varphi$

$z = -l\cos\theta$

$V = -mg\cos\theta\,,\qquad V = 0$ for $\theta = \frac{1}{2}\pi\,.$

↑ l




$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{l^2}{2}m\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)$$

$$L = T - V = \frac{l^2}{2}m\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) + mgl\cos\theta$$

Lagrangeligningen: $\dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\theta}} - \dfrac{\partial L}{\partial\theta} = 0$

Vi finner $\dfrac{\partial L}{\partial\dot{\theta}} = l^2 m\dot{\theta}$ og $\dfrac{\partial L}{\partial\theta} = l^2 m\sin\theta\cos\theta\dot{\varphi}^2 - mgl\sin\theta$

Det gir bevegelsesligningen: $\ddot{\theta} - \frac{1}{2}\sin 2\theta\cdot\dot{\varphi}^2 + \frac{g}{l}\sin\theta = 0$

Tilsvarende for $\varphi$: $\dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\varphi}} - \dfrac{\partial L}{\partial\varphi} = 0$, hvor $\dfrac{\partial L}{\partial\varphi} = 0$ som gir den andre

bevegelsesligningen: $\dfrac{d}{dt}\left(ml^2\sin^2\theta\cdot\dot{\varphi}\right) = 0\,,\qquad p_\varphi \equiv ml^2\sin^2\theta\,\dot{\varphi} = konst.$

Total energi er konstant: $E = T + V = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right) - mgl\cos\theta$

Innsetting av $\qquad \dot{\varphi} = \dfrac{p_\varphi}{ml^2\sin^2\theta}$

gir: $$E = \tfrac{1}{2}ml^2\dot{\theta}^2 + \frac{p_\varphi^{\,2}}{2ml^2\sin^2\theta} - mgl\cos\theta \equiv \tfrac{1}{2}ml^2\dot{\theta}^2 + V_{eff}(\theta),$$

hvor $$V_{eff}(\theta) = \frac{p_\varphi^{\,2}}{2ml^2\sin^2\theta} - mgl\cos\theta\,.$$ Altså $$\dot{\theta}^2 = \frac{2}{ml^2}\left(E - V_{eff}\right)$$

$$t = \int dt = \sqrt{\frac{ml^2}{2}}\int\frac{d\theta}{\sqrt{E - V_{eff}(\theta)}}$$

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Av $\dot{\varphi} = \dfrac{p_\varphi}{ml^2\sin^2(\theta)}$, følger

$$\varphi = \int\frac{p_\varphi}{ml^2\sin^2\theta}dt = \frac{p_\varphi}{\sqrt{2ml^2}}\int\frac{d\theta}{\sin^2\theta\sqrt{E - V_{eff}(\theta)}}$$

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$p_\varphi = 0$ gir $ml^2\sin^2\theta\dot{\varphi} = 0$, som har løsningen $\varphi = konst.$ som svarer til plan pendel.