CLASSICAL MECHANICS

1. Fundamental principles

1.1 Single particle

Position : $\vec{r}$

Velocity : $\vec{v} = \frac{d\vec{r}}{dt} \quad (= \dot{\vec{r}})$

Acceleration : $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \quad (= \dot{\vec{v}} = \ddot{\vec{r}})$

Linear momentum : $\vec{p} = m\vec{v}$; m = particle mass

Total force on particle : $\vec{F}$

Newton's 2. law (N2), valid in inertial frame: $\vec{F} = \frac{d\vec{p}}{dt}$

with constant mass m : $\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$

Conservation law for linear momentum:

$\vec{F} = 0 \Rightarrow \vec{p}$ is conserved

Angular momentum, relative to the origin:

$$\vec{L} = \vec{r} \times \vec{p}$$

Torque, relative to the origin:

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad (\text{GPS} : \vec{N})$$

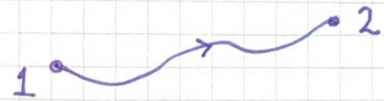
$$\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt} = \underbrace{\frac{d}{dt}(\vec{r} \times \vec{p})}_{=\vec{L}} - \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{=0} \Rightarrow \vec{\tau} = \frac{d\vec{L}}{dt}$$

Conservation law for angular momentum:

$\vec{\tau} = 0 \Rightarrow \vec{L}$ is conserved

Work done by force $\vec{F}$ when particle moves from 1 to 2:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s}$$



With constant m (assumed if otherwise not stated):

$$\vec{F} \cdot d\vec{s} = m \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt = m \cdot \frac{1}{2}\, d v^2$$

$$\Rightarrow W_{12} = \frac{1}{2} m \left(v_2^2 - v_1^2 \right) = T_2 - T_1$$

Kinetic energy of particle: $T = \frac{1}{2} m v^2$

Conservative system if W_{12} is independent of path from 1 to 2; then $\oint \vec{F} \cdot d\vec{s} = 0$, and we may write

$$\vec{F} = -\nabla V(\vec{r}) \quad ; \quad V = \text{potential energy}$$

(or simply potential)

Since $\nabla V = \nabla (V + \text{constant})$, the choice of $V = 0$ is arbitrary.

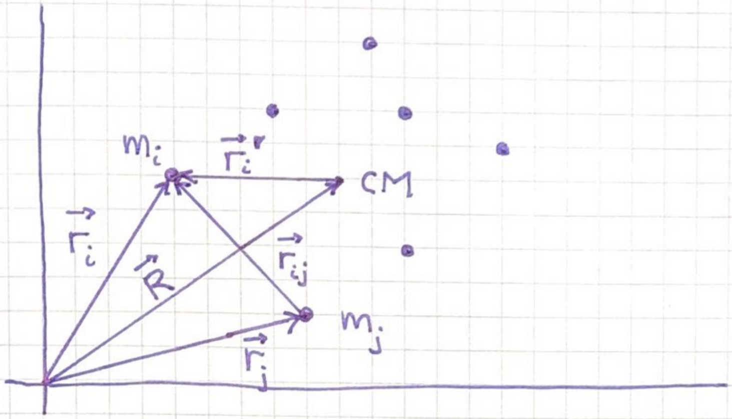
For cons. system:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s} = \int_2^1 \nabla V \cdot d\vec{s} = \int_2^1 dV = V_1 - V_2$$

$$\Rightarrow T_2 - T_1 = V_1 - V_2 \Rightarrow T_1 + V_1 = T_2 + V_2$$

Conservation law for particle energy:
In a conservative system, the total energy $T + V$ is conserved

1.2 System of particles



N2 for m_i : $\displaystyle\sum_{j\neq i} \vec{F}_{ji} + \vec{F}_i^{(e)} = \dot{\vec{p}}_i$

internal force from j on i

external force on i

Assume $\vec{F}_{ij}$ obeys N3, i.e., $\vec{F}_{ij} = -\vec{F}_{ji}$ (weak law of action/reaction) and sum over i :

$$\underbrace{\sum_{\substack{i,j\\ j\neq i}} \vec{F}_{ij}}_{=0} + \underbrace{\sum_i \vec{F}_i^{(e)}}_{=\vec{F}^{(e)} = \text{ total external force}} = \frac{d^2}{dt^2} \sum_i m_i \vec{r}_i$$

Center of mass (CM) : $\vec{R} = \dfrac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \dfrac{1}{M} \sum_i m_i \vec{r}_i$

$\Rightarrow M\ddot{\vec{R}} = \vec{F}^{(e)}$; CM moves as if all mass in CM

Total (linear) momentum : $\vec{P} = \sum_i m_i \dot{\vec{r}}_i = M\dot{\vec{R}}$

$\Rightarrow$ Conservation law : $\vec{F}^{(e)} = 0 \Rightarrow \vec{P}$ is conserved

Total angular momentum: $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$

$$\Rightarrow \dot{\vec{L}} = \sum_i \vec{r}_i \times \dot{\vec{p}}_i \qquad \left(\text{since } \sum_i \dot{\vec{r}}_i \times \vec{p}_i = 0\right)$$

$$\overset{N2}{=} \underbrace{\sum_i \vec{r}_i \times \vec{F}_i^{(e)}}_{\substack{= \text{ external} \\ \text{torque } \vec{\tau}^{(e)}}} + \sum_{\substack{i,j \\ i \neq j}} \vec{r}_i \times \vec{F}_{ji}$$

Double sum is sum of pairs:

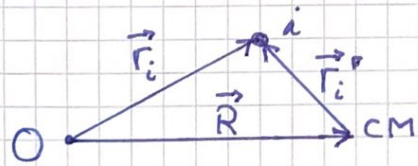
$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} \qquad (\text{since } \vec{F}_{ij} = -\vec{F}_{ji})$$

If internal forces are <u>central</u>, i.e., along the joining line $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ (strong law of action/reaction), all $\vec{r}_{ij} \times \vec{F}_{ji} = 0$

$$\Rightarrow \vec{\tau}^{(e)} = \dot{\vec{L}}$$

$\Rightarrow$ Conservation law: $\vec{\tau}^{(e)} = 0 \Rightarrow \vec{L}$ is conserved

Reformulation of $\vec{L}$:



$$\vec{r}_i = \vec{R} + \vec{r}_i'$$
$$\vec{v}_i = \vec{v} + \vec{v}_i'$$

$$\Rightarrow \vec{L} = \sum_i (\vec{R} + \vec{r}_i') \times m_i(\vec{v} + \vec{v}_i')$$

Since $\sum_i m_i \vec{r}_i' = 0$, two sums are zero

$$\Rightarrow \vec{L} = \underbrace{\vec{R} \times M\vec{v}}_{\substack{\text{orbital angular mom.} \\ \text{relative O}}} + \underbrace{\sum_i \vec{r}_i' \times \vec{p}_i'}_{\substack{\text{internal angular mom.} \\ \text{relative CM}}}$$

Work and kinetic energy :

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i$$

$$= \sum_i \int_1^2 m_i \dot{\vec{v}}_i \cdot \vec{v}_i \, dt = \sum_i \int_1^2 d\left(\tfrac{1}{2} m_i v_i^2\right)$$

$$= \left(\sum_i \tfrac{1}{2} m_i v_i^2\right)_2 - \left(\sum_i \tfrac{1}{2} m_i v_i^2\right)_1 = T_2 - T_1$$

Reformulation of T :

$$T = \frac{1}{2} \sum_i m_i \left(\vec{v} + \vec{v}_i'\right) \cdot \left(\vec{v} + \vec{v}_i'\right)$$

$$= \frac{1}{2} \sum_i m_i v^2 + \frac{1}{2} \sum_i m_i v_i'^2 + 2 \cdot \frac{\vec{v}}{2} \cdot \frac{d}{dt} \underbrace{\sum_i m_i \vec{r}_i'}_{=0}$$

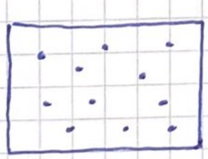
$$= \frac{1}{2} M v^2 + \frac{1}{2} \sum_i m_i v_i'^2$$

CM part

internal part
(e.g. rotation of rigid body)

1.3 Constraints and generalized coordinates

Examples of constrained motion :



gas in container

rigid body rolling on incline

Classification of constraints:

Holonomic : $f(\vec{r}_1, \vec{r}_2, \cdots, t) = 0$

Rheonomous : Time dependent

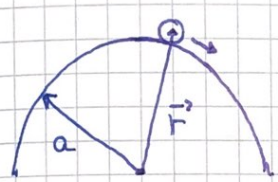
Scleronomous : Time independent

Examples :

Rolling on fixed incline : $y - ax - b = 0$
(holonomic, scleronomous)

Rolling on moving incline : holonomic, rheonomous

Rolling on spherical surface:



$r^2 - a^2 \geq 0 \Rightarrow$ nonholonomic

Generalized coordinates :

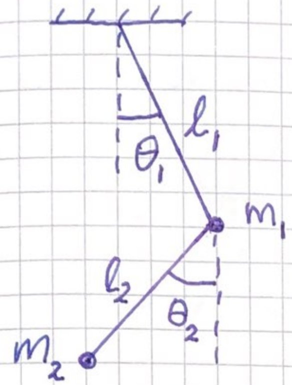
N particles has 3N independent coordinates, or degrees of freedom. With k holonomic constraints, the corresponding equations reduce this to 3N − k degrees of freedom.

New independent coordinates : $q_1, q_2, \cdots, q_{3N-k}$

Transformation equations, with constraints implicitly incorporated :

$$\vec{r}_j = \vec{r}_j(q_1, q_2, \cdots, q_{3N-k}, t) \quad ; \quad j = 1, 2, \cdots, N$$

Ex: Double pendulum (in a plane)



$N = 2$

$\Rightarrow 3N = 6$ degrees of freedom

Generalized coordinates:

θ_1 , θ_2

Constraints:

$z_1 = 0 \quad ; \quad z_2 = 0$ (movement in xy-plane)

$$\left.\begin{array}{l} x_1^2 + y_1^2 - l_1^2 = 0 \\ (x_2 - x_1)^2 + (y_2 - y_1^2) - l_2^2 = 0 \end{array}\right\} \text{(constant rod lengths)}$$

$\Rightarrow 6 - 4 = 2$ independent degrees of freedom

1.4 D'Alembert's Principle and Lagrange's Equations

We will show: For a conservative and holonomic system with n generalized coordinates q_j

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \qquad (j = 1, \dots, n)$$

$L = T - V$ = Lagrangian of the system

Proof:

Imagine infinitesimal displacements $\delta \vec{r}_i$ of the system coordinates, at a given time, and consistent with the constraints – a virtual displacement.

If equilibrium: $\vec{F}_i = 0 \quad ; \quad i = 1, \dots, N$

$$\Rightarrow \sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0 \quad \Rightarrow \sum_i \big(\underbrace{\vec{F}_i^{(a)}}_{\text{applied (external) force}} + \underbrace{\vec{f}_i}_{\text{force of constraint}} \big) \cdot \delta \vec{r}_i = 0$$

Assume $\sum_i \vec{f}_i \cdot \delta \vec{r}_i = 0$; not OK if $\vec{f}$ is sliding friction, but OK for frictionless sliding on a surface ($\vec{N} \perp \delta \vec{r}$) and rolling without sliding (contact point momentarily at rest).

$\Rightarrow$ We drop superscript (a), i.e., $\vec{F}_i$ = applied force. Constraint forces are eliminated, and the net virtual work is zero in equilibrium:

$$\sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0$$

Out of equilibrium : $\vec{F}_i - \dot{\vec{p}}_i = 0$

$$\Rightarrow \sum_i (\vec{F}_i - \dot{\vec{p}}_i)\cdot \delta\vec{r}_i = 0 \qquad \text{[D'Alembert's principle]}$$

Assume holonomic system and introduce independent coordinates q_j :

$$\vec{r}_i = \vec{r}_i\,(q_1, \dots, q_n, t) = \vec{r}_i\,(q_j, t)$$

Particle velocities :

$$\vec{v}_i = \dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} = \vec{v}_i\,(q_j, \dot{q}_j, t)$$

Virtual displacements :

$$\delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j \qquad [\text{Fixed time, } \delta t = 0]$$

1. sum :

$$\sum_i \vec{F}_i\cdot\delta\vec{r}_i = \sum_{i,j} \vec{F}_i\,\frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j = \sum_j Q_j\,\delta q_j$$

Generalized force : $Q_j = \sum_i \vec{F}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j}$

2. sum :

$$\sum_i \dot{\vec{p}}_i\cdot\delta\vec{r}_i = \sum_{i,j} m_i\ddot{\vec{r}}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j$$

Product rule $\Rightarrow$ $m_i\ddot{\vec{r}}_i\cdot\partial\vec{r}_i/\partial q_j = \frac{d}{dt}\left(m_i\vec{v}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i\vec{v}_i\cdot\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_j}$

Can $\frac{d}{dt}$ and $\frac{\partial}{\partial q_j}$ be interchanged in the last term? Yes :

$$\left.\begin{aligned}
\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_j} &= \sum_k\left(\frac{\partial}{\partial q_k}\frac{\partial \vec{r}_i}{\partial q_j}\right)\dot{q}_k + \frac{\partial}{\partial t}\frac{\partial \vec{r}_i}{\partial q_j}\\
\frac{\partial}{\partial q_j}\underbrace{\frac{d\vec{r}_i}{dt}}_{\vec{v}_i} &= \frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right)
\end{aligned}\right\}\ \text{OK!}$$

We also need $\dfrac{\partial \vec{v}_i}{\partial \dot{q}_j} = \dfrac{\partial \vec{r}_i}{\partial q_j}$ [Follows from $\vec{v}_i = \dots$ on p.9]

This yields:

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left\{ \frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right\}$$

$$= \sum_i \left\{ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left(\tfrac{1}{2} m_i v_i^2\right) - \frac{\partial}{\partial q_j} \left(\tfrac{1}{2} m_i v_i^2\right) \right\}$$

$$= \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \qquad \left[\text{since } T = \sum_i \tfrac{1}{2} m_i v_i^2\right]$$

D'Alembert's principle becomes:

$$\sum_j \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

With holonomic constraints and independent δq_j, each term in this sum must vanish:

$$\boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \quad ; \quad j = 1, 2, \cdots, n$$

If all forces are conservative, i.e., $\vec{F}_i = -\nabla_i V$:

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\left[\frac{\partial V}{\partial q_j} = \sum_i \left(\frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i} \frac{\partial z_i}{\partial q_j} \right) = \sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right]$$

We assume (for now) that V is independent of $\dot{q}_j$

$$\Rightarrow \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}(T - V)$$

$$\Rightarrow \boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad ; \quad L = T - V} \qquad \text{(qed)}$$

Scalar equations, <u>easier</u> than equations with vectors $\vec{F}_i$ and $\dot{\vec{p}}_i$

1.6 Simple examples of the Lagrangian formulation

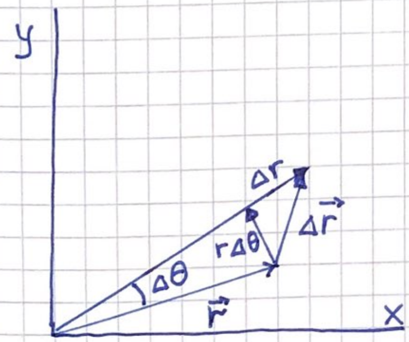
Ex. 1: Single particle

a) Cartesian coordinates

$$q_1 = x, \quad q_2 = y, \quad q_3 = z$$
$$Q_1 = F_x, \quad Q_2 = F_y, \quad Q_3 = F_z$$
$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$\frac{\partial T}{\partial q_j} = 0, \quad \frac{\partial T}{\partial \dot{q}_j} = m \dot{q}_j \quad ; \quad j = 1,2,3$$
$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \quad ; \quad -''-$$
$$\Rightarrow m\ddot{x} = F_x, \quad m\ddot{y} = F_y, \quad m\ddot{z} = F_z \quad \text{(N2, as expected)}$$

b) Plane polar coordinates

$$\left.\begin{array}{l} x = r \cos\theta \\ y = r \sin\theta \end{array}\right\} \Rightarrow \{q_j\} = \{r, \theta\}$$
$$\dot{x} = \dot{r} \cos\theta - r\dot{\theta} \sin\theta$$
$$\dot{y} = \dot{r} \sin\theta + r\dot{\theta} \cos\theta$$
$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$



$$\Delta\vec{r} = \Delta r\, \hat{r} + r \Delta\theta\, \hat{\theta}$$

Radial velocity: $\frac{\Delta r}{\Delta t} \to \dot{r}$

Azimuth $-''-$: $\frac{r\Delta\theta}{\Delta t} \to r\dot{\theta}$

Generalized force components : $Q_j = \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_j}$

$$Q_r = \vec{F} \cdot \frac{\partial \vec{r}}{\partial r} = \vec{F} \cdot \hat{r} = F_r$$

$$Q_\theta = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \theta} = \vec{F} \cdot r\hat{\theta} = r F_\theta$$

Lagrange's equations :

$$q_1 = r, \quad \frac{\partial T}{\partial r} = mr\dot{\theta}^2, \quad \frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{r}} = m\ddot{r}$$

$$\Rightarrow \quad m\ddot{r} - \underbrace{mr\dot{\theta}^2}_{\text{centripetal acceleration}} = F_r$$

$$q_2 = \theta, \quad \frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

$$\Rightarrow \quad mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = rF_\theta$$

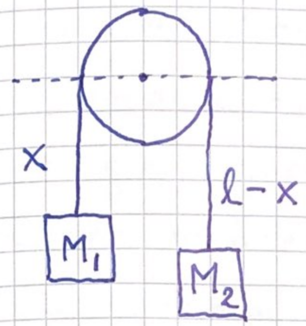
This is angular momentum, torque and N2 for rotation :

$$\vec{L} = \vec{r} \times \vec{p} \Rightarrow L = rp_\theta = rmv_\theta = rmr\dot{\theta} = mr^2\dot{\theta}$$

$$\vec{\tau} = \vec{r} \times \vec{F} \Rightarrow \tau = rF_\theta$$

$$\Rightarrow \quad \frac{d}{dt}(mr^2\dot{\theta}) = rF_\theta \quad \text{or} \quad \frac{dL}{dt} = \tau$$

Ex 2: Atwood's machine



$\Rightarrow$ only 1 independent coord: x

$$V = -M_1 g x - M_2 g (\ell - x)$$

$$T = \frac{1}{2}(M_1 + M_2)\dot{x}^2$$

$$L = T - V$$

$$\frac{\partial L}{\partial x} = (M_1 - M_2) g \,, \quad \frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\dot{x}$$

$$\Rightarrow \quad (M_1 + M_2)\ddot{x} - (M_1 - M_2) g = 0$$

$$\Rightarrow \quad \ddot{x} = \frac{M_1 - M_2}{M_1 + M_2} g$$

The constraint force, here the tension in the rope, does not appear in the Lagrangian formulation, as expected.

1.5 Generalized potentials

So far, we have assumed potentials $V(q_j)$ dependent on coordinates only. We would like to use the Lagrangian formulation also when the forces depend on the particle velocity, e.g., electromagnetism and fluid friction.

Electromagnetism:

First, note that Lagrange's equations have the same form if the generalized forces can be expressed by a generalized potential $U(q_j, \dot{q}_j)$ as

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j}$$

Then, with $L = T - U$ and $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$,

we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \qquad \text{as before.}$$

Maxwells equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ; \quad \nabla \cdot \vec{D} = \rho$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \quad ; \quad \nabla \cdot \vec{B} = 0$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \quad ; \quad \vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

In absence of magnetic or dielectric materials:

$\vec{M} = 0, \quad \vec{P} = 0$

$$\Rightarrow \nabla \cdot \vec{E} = \rho/\varepsilon_0 \quad ; \quad \nabla \times \vec{B} = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Lorentz force on moving charge: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Scalar potential ϕ and vector potential $\vec{A}$:

$\nabla \cdot \vec{B} = 0 \quad \Rightarrow$ may write $\vec{B} = \nabla \times \vec{A}$

(since $\nabla \cdot (\nabla \times \vec{V}) = 0$ for any vector $\vec{V}$)

$$\Rightarrow \nabla \times \vec{E} + \frac{\partial}{\partial t}(\nabla \times \vec{A}) = \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$\Rightarrow$ may write $\vec{E} + \partial \vec{A}/\partial t = -\nabla \phi$

(since $\nabla \times \nabla f = 0$ for any scalar f)

Lorentz force in terms of these potentials:

$$\vec{F} = q\left(-\nabla \phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \nabla \times \vec{A}\right)$$

Intermezzo on notation:

Einsteins summation convention:
A repeated index should implicitly be summed over all its possible values.

Assume cartesian coordinates with

$$x_1 = x, \quad x_2 = y, \quad x_3 = z$$

Then,

$$\begin{aligned} \vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y + A_z B_z \\ &= \sum_{i=1}^{3} A_i B_i \\ &= A_i B_i \end{aligned}$$

with summation over i assumed, without the Σ sign.

Other examples :

$$\nabla \cdot \vec{A} = \frac{\partial A_i}{\partial x_i} = \partial_i A_i$$

$$\nabla \phi = \hat{x}_i \partial_i \phi$$

Levi - Civita tensor :

Useful when dealing with cross products.

$$\varepsilon_{ijk} = +1 \quad \text{if } i,j,k \text{ cyclic} \quad (\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1)$$

$$-1 \quad \text{—"— anticyclic} \quad (\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1)$$

$$0 \quad \text{if 2 or 3 indices are equal}$$

Then:

$\vec{A} = \vec{B} \times \vec{C}$: $A_i = \varepsilon_{ijk} B_j C_k$

$$\text{Ex: } A_1 = \varepsilon_{123} B_2 C_3 + \varepsilon_{132} B_3 C_2$$
$$= B_2 C_3 - B_3 C_2 \qquad \text{(OK)}$$

$\vec{B} = \nabla \times \vec{A}$: $B_i = \varepsilon_{ijk} \partial_j A_k$

Often useful : $\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$

Ex: $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \times \vec{B})_i \, (\vec{C} \times \vec{D})_i$

$$= \varepsilon_{ijk} A_j B_k \varepsilon_{ilm} C_l D_m$$

$$= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) A_j B_k C_l D_m$$

$$= A_j C_j B_k D_k - A_j D_j B_k C_k$$

$$= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

[More in assignments.]

We had: $\vec{F} = q\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \nabla \times \vec{A}\right)$

$$\Rightarrow F_i = q\left\{-\partial_i \phi - \partial_t A_i + [\vec{v} \times (\nabla \times \vec{A})]_i\right\}$$

$$[\vec{v} \times (\nabla \times \vec{A})]_i = \varepsilon_{ijk} v_j \varepsilon_{klm} \partial_l A_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) v_j \partial_l A_m = v_j \partial_i A_j - v_j \partial_j A_i$$

1. term: $v_j \partial_i A_j = \partial_i v_j A_j = \partial_i \vec{v} \cdot \vec{A}$ (since $\partial_i v_j = 0$)

2. term: $v_j \partial_j A_i = \sum_j \frac{\partial A_i}{\partial x_j} \underbrace{\frac{dx_j}{dt}}_{v_j} = \frac{dA_i}{dt} - \frac{\partial A_i}{\partial t}$

$$\Rightarrow F_i = q\left\{-\partial_i \phi - \partial_t A_i + \partial_i \vec{v} \cdot \vec{A} - \left(\frac{dA_i}{dt} - \partial_t A_i\right)\right\}$$

$$= q\left\{-\partial_i [\phi - \vec{v} \cdot \vec{A}] + \frac{d}{dt}\frac{\partial}{\partial v_i}[\phi - \vec{v} \cdot \vec{A}]\right\}$$

(since $\partial \phi / \partial v_i = 0$ and $\partial (\vec{v} \cdot \vec{A}) / \partial v_i = A_i$)

Now, F_i is on the desired form,

$$F_i = -\frac{\partial U}{\partial x_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}_i}$$

with $U(x_i, \dot{x}_i) = q\phi - q\vec{A} \cdot \vec{v}$

Lagrangian for charged particle in E.M. field:

$$L = T - U = \frac{1}{2} m v^2 - q\phi + q\vec{A} \cdot \vec{v}$$

Friction:

May write Lagrange's equations on the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

with

$L = T - V$; V = potential due to conservative forces

Q_i = forces not expressed by a potential

Assume fluid friction and laminar flow, i.e., a frictional force prop. with the particle velocity:

$$F_x = -k_x v_x$$

May be expressed as $F_x = -\frac{\partial}{\partial v_x} \frac{1}{2} k_x v_x^2$, or in 3D $\vec{F} = -\nabla_v \mathcal{F}$ with

$$\mathcal{F} = \frac{1}{2}\sum_i \left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right)$$

= Rayleigh's dissipation function

Work done by the system (all particles i) against friction:

$$dW = -\sum_i \vec{F}_i \cdot d\vec{r}_i = -\sum_i \vec{F}_i \cdot \vec{v}_i \, dt = \sum_i \left(k_x^2 v_{ix}^2 + \dots\right) dt$$

$\Rightarrow dW/dt = 2\mathcal{F}$ = dissipated power due to friction

Generalized frictional force:

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \nabla_v \mathcal{F} \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

Ex: Stokes' law $\vec{F} = -6\pi\eta a \vec{v}$ for sphere of radius a moving in fluid with viscosity η