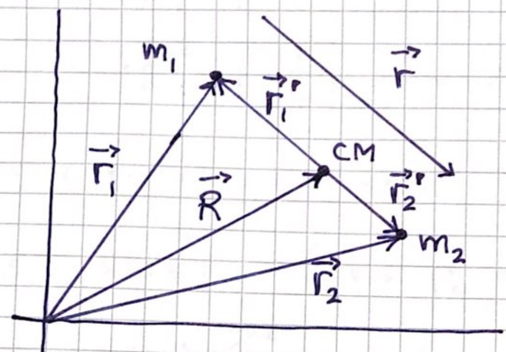
3. The Central Force Problem

3.1 Reduction to a one-body problem



$N = 2$

$\Rightarrow 3N = 6$

degrees of freedom

We choose as generalized coordinates (vectors)

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad \text{and} \quad \vec{r} = \vec{r}_2 - \vec{r}_1$$

Assume conservative system and central forces:

$$V = V(r) \quad ; \quad r = |\vec{r}|$$

Lagrangian : $L = T - V$

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} = \frac{1}{2} m_1 (\dot{\vec{R}} + \dot{\vec{r}}_1{}')^2 + \frac{1}{2} m_2 (\dot{\vec{R}} + \dot{\vec{r}}_2{}')^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + \underbrace{\frac{1}{2} m_1 \dot{\vec{r}}_1{}'^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2{}'^2}_{T'} + \dot{\vec{R}} \cdot (m_1 \dot{\vec{r}}_1{}' + m_2 \dot{\vec{r}}_2{}')$$

Clearly, $m_1 \vec{r}_1{}' + m_2 \vec{r}_2{}' = 0$ ("CM relative to CM")

$$\Rightarrow \frac{d}{dt}(m_1 \vec{r}_1{}' + m_2 \vec{r}_2{}') = 0$$

$$\Rightarrow T = \frac{1}{2} M \dot{\vec{R}}^2 + T' \quad ; M = m_1 + m_2$$

Also, $\vec{r} = \vec{r}_2{}' - \vec{r}_1{}' \quad \Rightarrow \quad \vec{r}_1{}' = -\frac{m_2 \vec{r}}{M}, \quad \vec{r}_2{}' = \frac{m_1 \vec{r}}{M}$

$$\Rightarrow T' = \frac{m_1 m_2}{2M} \dot{\vec{r}}^{\,2}$$

The reduced mass: $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \Rightarrow \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$

$$\Rightarrow T' = \frac{1}{2}\mu \dot{\vec{r}}^{\,2}$$

$$\Rightarrow L = \frac{1}{2} M \dot{\vec{R}}^{2} + \frac{1}{2}\mu \dot{\vec{r}}^{\,2} - V(r) = L(r, \dot{\vec{r}}, \dot{\vec{R}})$$

$\Rightarrow \vec{R}$ is a cyclic coordinate

$\Rightarrow \dot{\vec{R}}$ is constant (or: $p_R = \frac{\partial L}{\partial \dot{R}} = M\dot{R} = \text{const.}$)

$\Rightarrow$ CM term $\frac{1}{2} M \dot{\vec{R}}^{2}$ can be ignored

$\Rightarrow$ Two-body problem is reduced to one-body problem, with Lagrangian

$$L = T - V = \frac{1}{2}\mu \dot{\vec{r}}^{\,2} - V(r)$$

3.2 Equations of motion

So, we consider a mass m in a central forcefield, with $V = V(r)$. Rotational invariance (around any axis) $\Rightarrow$ Angular momentum is conserved:

$$\vec{L} = \vec{r} \times \vec{p} = \text{constant}$$

Only possible if $\vec{r}$ and $\vec{p}$ lie in a plane $\perp \vec{L}$

$\Rightarrow$ We use polar coordinates (r, θ).

$$\Rightarrow L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

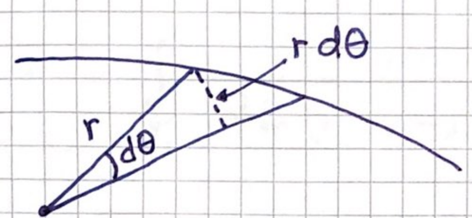
$$= L(r, \dot{r}, \dot{\theta}) \qquad [\text{cf Ex 1b, p 11}]$$

L independent of θ (θ is cyclic coordinate)

$$\Rightarrow \quad p_\theta = \partial L / \partial \dot\theta = m r^2 \dot\theta \quad \text{is conserved}$$

$$= \ell = \text{abs. value of the angular momentum}$$

Kepler's 2. law:



$$dA = \tfrac{1}{2} r \cdot r\,d\theta$$

$$\Rightarrow \dot A = \tfrac{1}{2} r^2 \dot\theta$$

$$= \tfrac{1}{2m} p_\theta = \text{constant}$$

The radius r sweeps across a constant area pr unit time

Eqn. of motion for r:

$$\frac{\partial L}{\partial r} = m r \dot\theta^2 - \frac{\partial V}{\partial r} \; ; \quad \frac{\partial L}{\partial \dot r} = m \dot r \; ; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot r} = m \ddot r$$

$$= \frac{\ell^2}{m r^3} + f(r) \qquad (\text{using } \dot\theta = \ell / m r^2)$$

$$\Rightarrow \quad m \ddot r - \frac{\ell^2}{m r^3} = f(r)$$

$$f(r) = -\partial V / \partial r = \text{force in radial direction } \hat r$$

Conservative system $\Rightarrow$ also total energy E is conserved,

$$E = T + V = \tfrac{1}{2} m \left(\dot r^2 + \overbrace{r^2 \dot\theta^2}^{\ell^2/m^2 r^2} \right) + V(r)$$

Two 2. order eqns. have been transformed into two 1. order eqns. using conservation of ℓ and E.

Solution for $r(t)$:

$$\frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V = E$$

$$\Rightarrow \frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\left(E - V - \ell^2/2mr^2\right)}$$

Assume $r(t=0) = r_0$ = minimum radius :

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}(E - V - \ell^2/2mr^2)}} = t(r;\ E, \ell, r_0)$$

$\Rightarrow$ $r(t)$ by inverting $t(r)$

Solution for $\theta(t)$:

$$\dot{\theta} = \ell/mr^2 \Rightarrow d\theta = \ell\, dt/mr^2(t)$$

$$\Rightarrow \theta(t) = \ell \int_0^t \frac{dt}{mr^2(t)} + \theta_0 \quad ; \quad \theta_0 = \theta(t=0)$$

Integration constants : E, ℓ, r_0, θ_0 (4, as expected)

May choose $r_0, \theta_0, \dot{r}_0, \dot{\theta}_0$ in classical mechanics, but better with E and ℓ e.g. when comparing classical with quantum mechanics since initial values for all four would violate Heisenberg's uncertainty relations.

3.3 Equivalent onedimensional problem

General insight from equation for r viewed as a onedimensional problem. We had (p. 38):

$$m\ddot{r} = f(r) + \frac{\ell^2}{mr^3} = f'(r)$$

i.e. N2 for 1D problem with mass m and force

$$f' = f + \ell^2/mr^3$$

Centrifugal force:

$$\ell^2/mr^3 = (mr^2\dot{\theta})^2/mr^3 = mr\dot{\theta}^2 = m v_\theta^2/r$$

Energy considerations; we had (p. 39):

$$E = \frac{1}{2}m\dot{r}^2 + \ell^2/2mr^2 + V(r)$$

i.e. 1D problem with potential

$$V'(r) = V(r) + \ell^2/2mr^2$$

and corresponding force

$$f' = -\partial V'/\partial r = -\partial V/\partial r + \ell^2/mr^3$$

$$= f + \ell^2/mr^3$$

as above.

Ex 1. Attractive $\frac{1}{r^2}$ force

$$f(r) = -k/r^2 \quad \Rightarrow \quad V(r) = -k/r$$

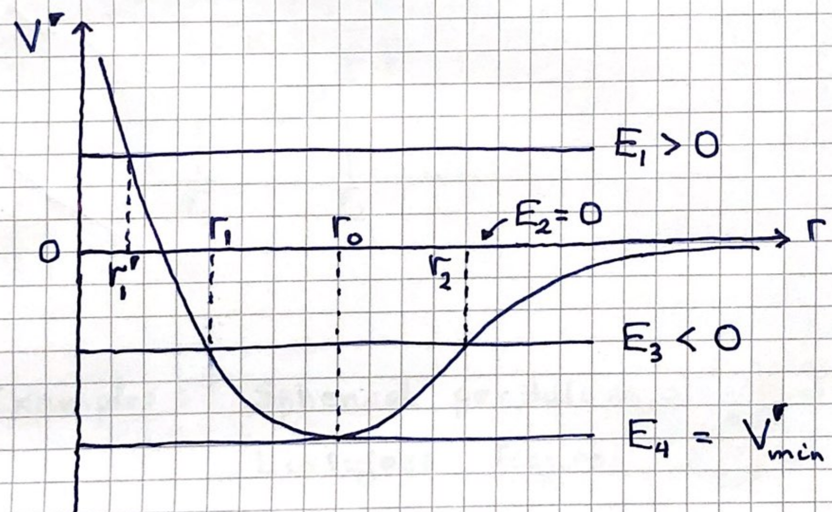
$$\Rightarrow V'(r) = -\frac{k}{r} + \frac{\ell^2}{2mr^2} \longrightarrow \begin{cases} +\infty & ; \; r \to 0 \\ -0 & ; \; r \to \infty \end{cases}$$

Centrifugal barrier : $\ell^2/2mr^2$

[Cf. Coulomb potential in QM, with quantized $\vec{L}^2$,

$$V^{\ell}_{eff}(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{\ell(\ell+1)\hbar^2}{2mr^2} \quad ; \quad \ell = 0, 1, 2, \ldots$$]

$$E = \frac{1}{2} m\dot{r}^2 + V'(r) \geq V'(r)$$



$E_1 > 0$: unbounded motion, $r \geq r_1'$ (hyperbola)

$E_2 = 0$: —— " —— (parabola)

$E_3 < 0$: bounded motion, $r_1 \leq r \leq r_2$ (ellipse)

$E_4 = V'_{min}$: —— " —— , $r = r_0$ (circle)

As 1D problem: m at rest, force f balances centrifugal force

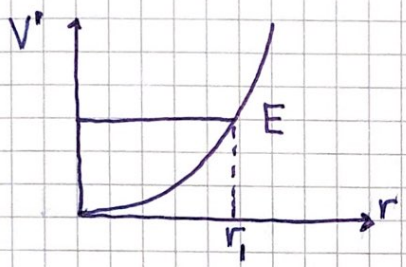
More about the Kepler problem and orbits in 3.7.

Ex. 2: Isotropic harmonic oscillator

$$f(r) = kr \; ; \quad V(r) = \frac{1}{2}kr^2 \; ; \quad V'(r) = \frac{1}{2}kr^2 + \frac{\ell^2}{2mr^2}$$

$\ell = 0$: (~~Null dreieimpuls~~) Zero angular momentum

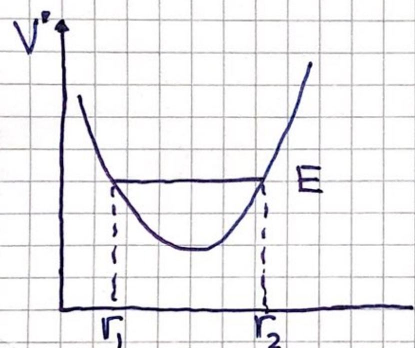
$\Rightarrow \vec{r} \parallel \vec{v} \quad \Rightarrow$ m moves along a straight line



$V' = V = \frac{1}{2}kr^2$

Bounded harmonic motion with $r \leq r_1$

$\ell > 0$:



$V' = \frac{1}{2}kr^2 + \frac{\ell^2}{2mr^2}$

Bounded motion with

$r_1 \leq r \leq r_2$

Examples: Spherical pendulum

Lissajous figures

(see assignments)

3.4 The virial theorem

Assume system of point masses m_i in positions $\vec{r}_i$, influenced by forces $\vec{F}_i$. Consider the function

$$G = \sum_i \vec{p}_i \cdot \vec{r}_i \quad \text{and} \quad \dot{G} = \sum_i \dot{\vec{r}}_i \cdot \vec{p}_i + \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i$$

1. sum: $\sum_i \dot{\vec{r}}_i \cdot m_i \dot{\vec{r}}_i = \sum_i m_i v_i^2 = 2T$

2. sum: $\sum_i \dot{\vec{p}}_i \cdot \vec{r}_i \overset{N2}{=} \sum_i \vec{F}_i \cdot \vec{r}_i$

$$\Rightarrow \quad \dot{G} = 2T + \sum_i \vec{F}_i \cdot \vec{r}_i$$

Take time average over time interval τ:

$$\overline{\dot{G}} = \frac{1}{\tau} \int_0^\tau \frac{dG}{dt}\, dt = \frac{1}{\tau}\left[G(\tau) - G(0)\right]$$

Assume long time interval: $\lim_{\tau \to \infty} \overline{\dot{G}} = 0$
(or periodic motion with period τ)

$$\Rightarrow \quad \overline{T} = -\frac{1}{2} \overline{\sum_i \vec{F}_i \cdot \vec{r}_i} \qquad \underline{\text{Virial theorem}}$$

Ex: Single particle in conservative central field

$$\overline{T} = \frac{1}{2} \overline{(\partial V / \partial r) \cdot r} \qquad \text{since} \quad \vec{F} = -\hat{r} \frac{\partial V}{\partial r}$$

Assume $V(r) = a\, r^{n+1}$; then $r\, \partial V / \partial r = (n+1) V$

$$\Rightarrow \quad \overline{T} = \frac{n+1}{2} \overline{V}$$

Harmonic oscillator: $n = 1 \Rightarrow \overline{T} = \overline{V}$

Gravitational field: $n = -2 \Rightarrow \overline{T} = -\frac{1}{2} \overline{V}$

3.7 The Kepler problem

We will find the analytical solution $r(\theta)$ when $f(r) = -k/r^2$ and $V(r) = -k/r$ (Ex 1, p. 41). We had:

$$\frac{1}{2} m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V = E \quad \text{and} \quad \dot{\theta} = \frac{\ell}{mr^2}$$

$$\Rightarrow \quad dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \ell^2/2mr^2\right)}} \quad \text{and} \quad dt = \frac{mr^2}{\ell}\, d\theta$$

$$\Rightarrow \quad d\theta = \frac{\ell\, dr}{mr^2 \sqrt{\frac{2}{m}\left(E - V - \ell^2/2mr^2\right)}}$$

$$\Rightarrow \quad \theta = \int_{r_0}^{r} \frac{dr}{r^2 \sqrt{\frac{2mE}{\ell^2} + \frac{2mk}{r\ell^2} - \frac{1}{r^2}}} + \theta_0$$

Choose $\theta_0 = \theta(r_0) = 0$ in perihel, where the particle is nearest the origin.

Substitute $u = 1/r \Rightarrow du = -dr/r^2$:

$$\theta = -\int_{u_0}^{u} du \left(\frac{2mE}{\ell^2} + \frac{2mku}{\ell^2} - u^2\right)^{-1/2}$$

"Standard integral":

$$\int dx \left(\alpha + \beta x - x^2\right)^{-1/2} = \arccos \frac{2x - \beta}{\sqrt{4\alpha + \beta^2}} = \arccos \frac{\frac{2x}{\beta} - 1}{\sqrt{1 + \frac{4\alpha}{\beta^2}}}$$

$$\Rightarrow \quad \theta = -\arccos \frac{\ell^2/mkr - 1}{\sqrt{1 + 2E\ell^2/mk^2}} = -\arccos \frac{p/r - 1}{\varepsilon}$$

$$\Rightarrow \quad \cos\theta = \cos(-\theta) = \frac{p/r - 1}{\varepsilon} \quad \Rightarrow \quad r = \frac{p}{1 + \varepsilon \cos\theta}$$

Here, we have introduced

the eccentricity : $\varepsilon = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$

the "parameter of the path" : $p = \ell^2/mk$

The equation $r = p/(1+\varepsilon\cos\theta)$ describes a conic section with focal point (focus) in the origin. Notice that $\theta = 0$ corresponds to $r_{min} = p/(1+\varepsilon)$; perihel.

Classification of various types of paths:

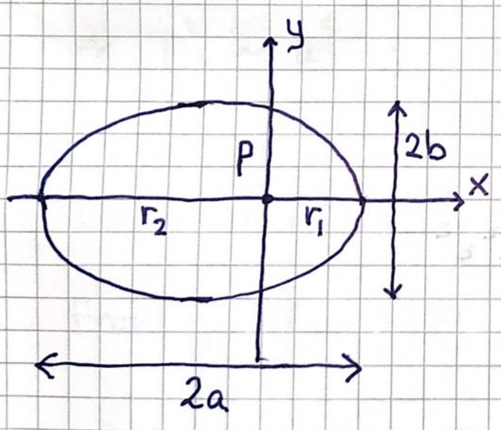
$\varepsilon > 1$: hyperbola $(E > 0)$

$\varepsilon = 1$: parabola $(E = 0)$

$\varepsilon < 1$: ellipse $(E < 0)$

$\varepsilon = 0$: circle $(E = -mk^2/2\ell^2)$

Elliptic orbit:



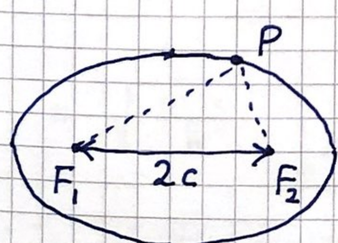
semi-major axis : a

semi-minor axis : b

$$2a = r_1 + r_2 = \frac{p}{1+\varepsilon} + \frac{p}{1-\varepsilon} = \frac{2p}{1-\varepsilon^2}$$

$$\Rightarrow a = \frac{p}{1-\varepsilon^2} = -\frac{k}{2E} = \frac{k}{2|E|} \quad (E<0)$$

Focal points:



$PF_1 + PF_2 = 2a$

$\varepsilon = c/a$

$a^2 = b^2 + c^2$

$$\Rightarrow b^2 = a^2 - c^2 = a^2(1-\varepsilon^2) = \left[\frac{p}{1-\varepsilon^2}\right]^2 (1-\varepsilon^2)$$
$$= \frac{p^2}{1-\varepsilon^2}$$

$$\Rightarrow b = \frac{p}{\sqrt{1-\varepsilon^2}} = \frac{\ell^2/mk}{\sqrt{-2E\ell^2/mk^2}} = \frac{\ell}{\sqrt{-2mE}} = \frac{\ell}{\sqrt{2m|E|}}$$

Period :

From p. 38 : $\ell = mr^2\dot{\theta}$, $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{\ell}{2m}$

$$\Rightarrow A = \int dA = \int_0^T \frac{dA}{dt}\,dt = \frac{\ell}{2m}\cdot T$$

Ellipse : $A = \pi ab$

$a = k/2|E| \Rightarrow 1/\sqrt{2|E|} = \sqrt{a/k} \Rightarrow \frac{b}{\ell} = \sqrt{\frac{a}{m\cdot k}}$

$$\Rightarrow T = 2\pi m\cdot a\cdot\frac{b}{\ell} = 2\pi\sqrt{\frac{m}{k}}\cdot a^{3/2}$$

$\Rightarrow T^2 \sim a^3$; Kepler's 3. law

The motion in time :

From p. 39, with $V = -k/r$:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}\left(E + k/r - \ell^2/2mr^2\right)}} = \sqrt{\frac{m}{2|E|}}\int\frac{r\,dr}{\sqrt{-r^2 + \frac{kr}{|E|} - \frac{\ell^2}{2m|E|}}}$$

$$= \sqrt{\frac{m\cdot a}{k}}\int\frac{r\,dr}{\sqrt{-r^2 + 2ar - b^2}}$$

Since $-b^2 = -a^2 + a^2\varepsilon^2$, we have

$$-r^2 + 2ar - b^2 = -(r-a)^2 + a^2\varepsilon^2$$

Substitute $r = a - a\varepsilon\cos\xi$

Then: $r \cdot dr = (a - a\varepsilon\cos\xi) \cdot a\varepsilon\sin\xi \cdot d\xi$

$$\sqrt{-r^2+2ar-b^2} = \sqrt{a^2\varepsilon^2 - a^2\varepsilon^2\cos^2\xi} = a\varepsilon\sin\xi$$

$$\Rightarrow t = \sqrt{\frac{ma^3}{k}} \int (1-\varepsilon\cos\xi)\,d\xi = \sqrt{\frac{ma^3}{k}}\,(\xi - \varepsilon\sin\xi)$$

where we chose $t=0$ for $\xi = 0$. Then, at $t=0$,

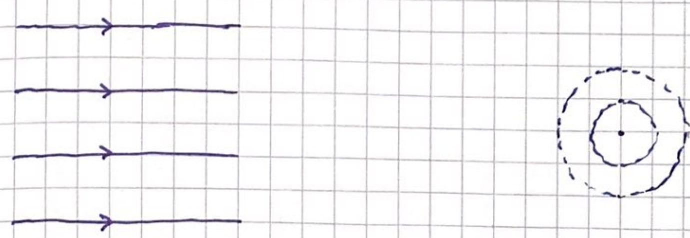
$$r = a(1-\varepsilon) = r_{min} \quad \text{(perihel)}$$

since $r_{min} = p/(1+\varepsilon)$, $a = p/(1-\varepsilon^2)$ $\Rightarrow$ $\frac{r_{min}}{a} = 1-\varepsilon$

Comments:

- $\xi = 2\pi \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}\,a^{3/2}$, as before
- $\cos\theta = (\cos\xi - \varepsilon)/(1-\varepsilon\cos\xi) \Rightarrow \theta = 0$ at $t=0$ (OK)
  (See textbook for details)

3.10 Scattering in a central force field



Central forcefield, attractive or repulsive.

$$f(r) = -\frac{\partial V}{\partial r}$$

$$\lim_{r\to\infty} f(r) = 0$$

Uniform flow of particles with equal mass m and energy E.

I = intensity = # particles pr unit time, and unit area ⊥ particle flow

$\sigma(\Omega)$ = differential scattering cross section

$\sigma(\Omega)\,d\Omega$ = (# particles scattered into solid angle $d\Omega$ pr unit time) / I

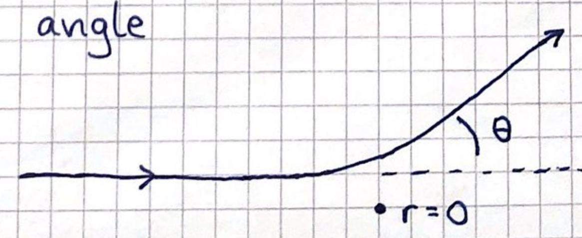
$$d\Omega = \sin\theta \, d\theta \, d\varphi$$

$$[\sigma] = s^{-1} / s^{-1} m^{-2} = m^2$$

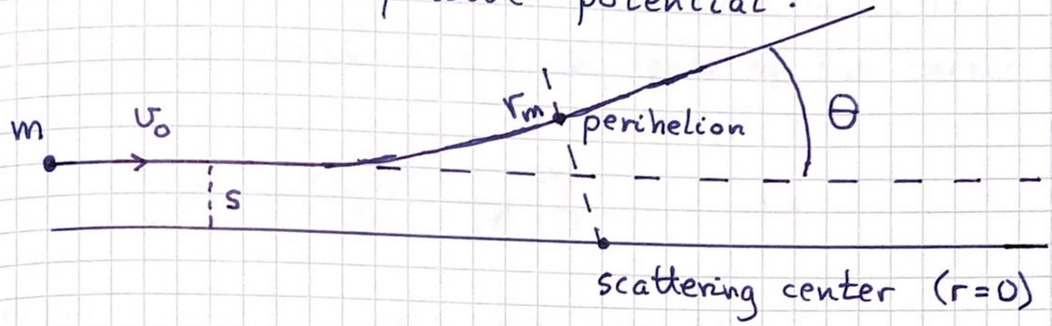
Central force $\Rightarrow$ Symmetry around incoming axis

$\Rightarrow d\Omega = 2\pi \sin\theta \, d\theta$

θ = scattering angle



Assume a repulsive potential:



s = impact parameter

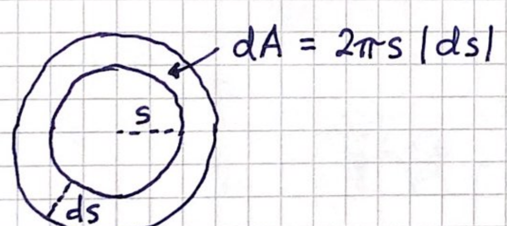
$$\ell = \lim_{r\to\infty} |\vec{r} \times \vec{p}| = m v_0 s = m\cdot\sqrt{\frac{2E}{m}}\cdot s = s\cdot\sqrt{2mE}$$

$\theta = \theta(s)$; $\theta(s=0) = \pi$; $\theta(s\to\infty) = 0$

θ decreases steadily with increasing s (for repulsive V)

$\Rightarrow$ # particles in between s and $s+ds$

= # particles out between θ and $\theta + d\theta$



$$\Rightarrow \quad I\cdot \underbrace{2\pi s\,|ds|}_{dA} = I\cdot \sigma(\theta)\cdot \underbrace{2\pi \sin\theta\cdot|d\theta|}_{d\Omega}$$

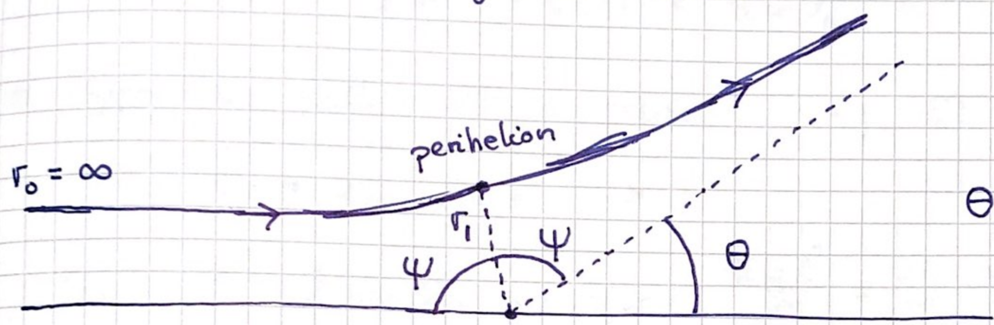
[Here, $ds/d\theta < 0$, hence $|ds|$ and $|d\theta|$]

$$\Rightarrow \quad \sigma(\theta) = \frac{s}{\sin\theta}\cdot\left|\frac{ds}{d\theta}\right|$$

We need $s(\theta)$. Start from integral on p. 44 with β as angle along the path of the particle:

$$\beta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}} + \beta_0$$

Geometry and integration limits:



$\theta + 2\Psi = \pi$

$$r_0 = \infty \quad \Rightarrow \quad \beta_0 = \pi$$

$$r = r_1 \quad \Rightarrow \quad \beta = \pi - \Psi$$

$$\Rightarrow \quad \pi - \Psi = \int_{\infty}^{r_1} \frac{dr}{r^2\sqrt{\cdots\cdots}} + \pi$$

$$\Rightarrow \quad \Psi = \int_{r_1}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}}$$

Now, introduce s:

$$\ell = s\sqrt{2mE} \quad \Rightarrow \quad \frac{2mE}{\ell^2} = \frac{1}{s^2} \quad ; \quad \frac{2mV}{\ell^2} = \frac{V}{E\cdot s^2}$$

$$\Rightarrow \Psi = \int_{r_1}^{\infty} \frac{s\,dr}{r^2\sqrt{1 - \frac{V}{E} - s^2/r^2}}$$

or with $u = 1/r$, $du = -dr/r^2$, $u_1 = 1/r_1$; $\theta = \pi - 2\Psi$:

$$\theta(s) = \pi - 2\int_0^{u_1} \frac{s\cdot du}{\sqrt{1 - V/E - s^2u^2}}$$

Ex. with analytical solution:

$m \bullet \longrightarrow v_0$

$q' = Z'e$

$M \gg m$ (Scattering center and mass center)

$q = Ze$

Coulomb potential: $V = -\frac{k}{r}$; $k = -ZZ'e^2/4\pi\varepsilon_0 < 0$

$$E = T + V = \frac{1}{2}mv^2 + V(r) = \frac{1}{2}mv_0^2 > 0$$

$\Rightarrow$ Hyperbolic path with eccentricity (p. 45)

$$\varepsilon = \sqrt{1 + 2E\ell^2/mk^2} > 1$$

Since $\ell^2 = s^2 \cdot 2mE$, we have $\frac{2E\ell^2}{m} = \frac{2E}{m} \cdot s^2 \cdot 2mE = 4E^2s^2$

$$\Rightarrow \quad \varepsilon = \sqrt{1 + 4E^2s^2/k^2}$$

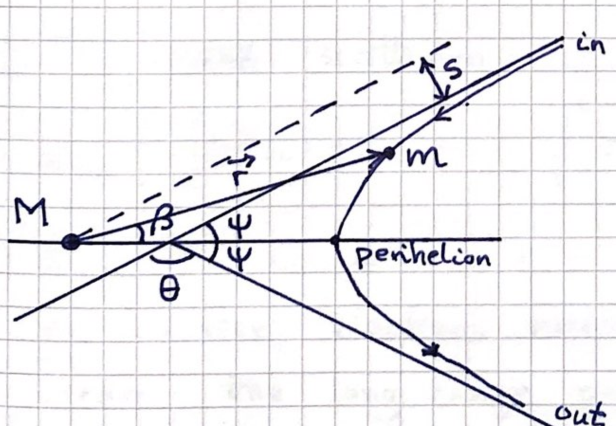
Parameter of the path: $p = \ell^2/mk < 0$

The derivation on p. 44 now yields the result

$$\cos\beta = \frac{1 - p/r}{\varepsilon} = \frac{1 + |p|/r}{\varepsilon}$$

or $|p|/r = \varepsilon\cos\beta - 1$ or $\underline{r = \frac{|p|}{\varepsilon\cos\beta - 1}}$

for the conic section.



Asymptotes:

$r \to \infty$

$\Rightarrow \beta \to (\pm)\,\psi$

$\Rightarrow \varepsilon\cos\psi = 1$

$\Rightarrow \cos\psi = 1/\varepsilon$

Scattering angle: $\theta = \pi - 2\psi \Rightarrow \psi = \frac{\pi}{2} - \frac{\theta}{2}$

$$\Rightarrow \cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \sin\frac{\theta}{2} = \frac{1}{\varepsilon}$$

$$\Rightarrow \varepsilon^2 - 1 = \frac{1}{\sin^2\theta/2} - \frac{\sin^2\theta/2}{\sin^2\theta/2} = \frac{\cos^2\theta/2}{\sin^2\theta/2} = \underline{\cot^2\theta/2}$$

$\parallel$ (p.51)

$\underline{4E^2 s^2/k^2}$

$$\Rightarrow s(\theta) = \frac{|k|}{2E}\cot\frac{\theta}{2}$$

[sign check: $0<\theta<\pi \Rightarrow$
$0<\frac{\theta}{2}<\frac{\pi}{2} \Rightarrow \cot\frac{\theta}{2}>0$]

$$\Rightarrow \left|\frac{ds}{d\theta}\right| = \frac{|k|}{4E}\cdot\frac{1}{\sin^2(\theta/2)}$$

$$\Rightarrow \sigma(\theta) = \frac{s}{\sin\theta}\cdot\left|\frac{ds}{d\theta}\right| = \frac{k^2}{8E^2}\cdot\frac{\cot\frac{\theta}{2}}{\sin\theta\cdot\sin^2\theta/2}$$

$\cot\theta/2 = \cos(\theta/2)/\sin(\theta/2)$

$\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$

$$\Rightarrow \underline{\sigma(\theta) = \left(\frac{k}{4E}\right)^2\cdot\frac{1}{\sin^4(\theta/2)}}$$

The Rutherford scattering cross section

Comments:

- Surprisingly, nonrelativistic quantum mechanics gives exactly the same result.
- The <u>total</u> scattering cross section <u>diverges</u>:

$$\sigma_{TOT} = \int\sigma(\Omega)\,d\Omega = 2\pi\int_0^\pi \sigma(\theta)\sin\theta\,d\theta \rightarrow \infty$$

I.e., the Coulomb force is Long range.
In practice, electrons surrounding the nucleus "screen" the long range potential and results in a finite total cross section.