4. Rigid body kinematics

4.1 Independent coordinates

N particles $\Rightarrow$ 3N degrees of freedom

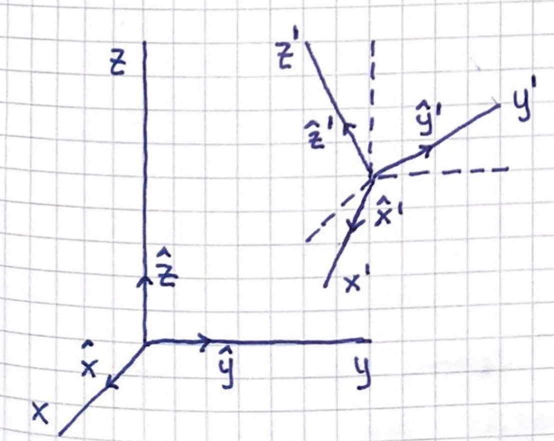
Holonomic constraints : $r_{ij} = c_{ij}$ $\quad [\frac{1}{2} N(N-1)$ constraints$]$

$\Rightarrow$ Only 6 degrees of freedom :

- 3 coords. to specify a reference point in the rigid body (e.g. CM)
- 3 $\overbrace{\text{coords.}}^{\text{(angles)}}$ to specify orientation of the rigid body relative to the chosen reference frame

Alternatively:
6 independent movements, translation along 3 axes and rotation around 3 axes.

Let x, y, z be axes in a fixed (external) coord. system, and x', y', z' axes in a coord. system fixed in the rigid body :



3 coords. specify the origin in the (x' y' z') system.

$\Rightarrow$ Need 3 more to specify the directions of x', y', z' relative to x, y, z

Direction cosines :

$$\alpha_1 = \cos(\hat{x}', \hat{x}) = \hat{x}' \cdot \hat{x}$$
$$\alpha_2 = \cos(\hat{x}', \hat{y}) = \hat{x}' \cdot \hat{y}$$
$$\alpha_3 = \cos(\hat{x}', \hat{z}) = \hat{x}' \cdot \hat{z}$$

Similarly with β for $\hat{y}'$ and γ for $\hat{z}'$.
The direction cosines are the components of $\hat{x}', \hat{y}', \hat{z}'$ along the external axes $\hat{x}, \hat{y}, \hat{z}$:

$$\hat{x}' = \alpha_1 \hat{x} + \alpha_2 \hat{y} + \alpha_3 \hat{z}$$
$$\hat{y}' = \beta_1 \hat{x} + \beta_2 \hat{y} + \beta_3 \hat{z}$$
$$\hat{z}' = \gamma_1 \hat{x} + \gamma_2 \hat{y} + \gamma_3 \hat{z}$$

And the other way around :

$$\hat{x} = \alpha_1 \hat{x}' + \beta_1 \hat{y}' + \gamma_1 \hat{z}'$$
$$\hat{y} = \alpha_2 \hat{x}' + \beta_2 \hat{y}' + \gamma_2 \hat{z}'$$
$$\hat{z} = \alpha_3 \hat{x}' + \beta_3 \hat{y}' + \gamma_3 \hat{z}'$$

6 constraints reduce 9 direction cosines to 3 independent coordinates :

$$\hat{x} \cdot \hat{x} = (\alpha_1 \hat{x}' + \beta_1 \hat{y}' + \gamma_1 \hat{z}')^2 = \alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1$$

$$\hat{x} \cdot \hat{y} = \alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2 = 0$$

etc.

$$\Rightarrow \quad \alpha_i \alpha_j + \beta_i \beta_j + \gamma_i \gamma_j = \delta_{ij}$$

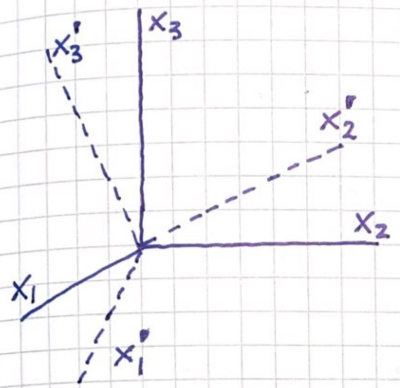
Soon : Euler angles, 3 independent functions of the direction cosines

4.2 Orthogonal transformations

Focus on rotation.

Assume common origin in external and body system.

Notation: $x, y, z \rightarrow x_1, x_2, x_3$ (more convenient)



Coord. transformations:

$$x_1' = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$
$$x_2' = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$
$$x_3' = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3$$

i.e., a linear transf.

General notation:

$$x_1' = a_{11} x_1 + a_{12} x_2 + a_{13} x_3$$
$$x_2' = a_{21} x_1 + a_{22} x_2 + a_{23} x_3$$
$$x_3' = a_{31} x_1 + a_{32} x_2 + a_{33} x_3$$

With summation convention:

$$x_i' = a_{ij} x_j \quad ; \quad i = 1, 2, 3 \qquad (1)$$

A rotation leaves $r = |\vec{r}|$ unchanged:

$$x_i' x_i' = x_i x_i \Rightarrow a_{ij} x_j a_{ik} x_k = x_i x_i$$

$$\Rightarrow a_{ij} a_{ik} = \delta_{jk} \qquad (2)$$

i.e., the 6 constraints on p. 54.

When (2) are satisfied, (1) is called an orthogonal transformation, and (2) is the orthogonality condition.

Transformation matrix:

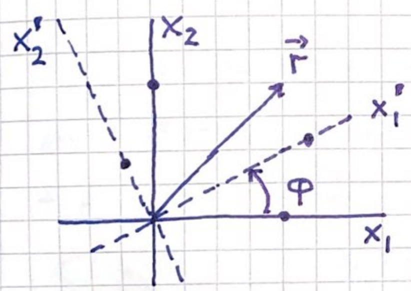
$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$ with matrix elements a_{ij}

Ex: Rotation in 2D

$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad ; \quad a_{ij}\, a_{ik} = \delta_{jk}$$

4 matrix elements, 3 orthog. conditions

$\Rightarrow$ 1 independent coordinate, the rotation angle φ, OK!



From the figure (check yourself):

$$x_1' = x_1 \cos\varphi + x_2 \sin\varphi$$
$$x_2' = -x_1 \sin\varphi + x_2 \cos\varphi$$

$\Rightarrow a_{11} = a_{22} = \cos\varphi \; ; \; a_{12} = -a_{21} = \sin\varphi$

$$\Rightarrow \mathbb{A} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

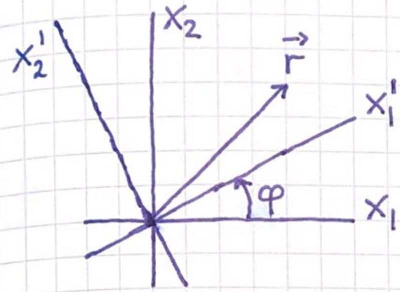
Orthog. conditions:

$$\left.\begin{array}{lll} j=k=1: & a_{11}a_{11} + a_{21}a_{21} = 1 & \Rightarrow \cos^2\varphi + \sin^2\varphi = 1 \\ j=k=2: & a_{12}a_{12} + a_{22}a_{22} = 1 & \Rightarrow \sin^2\varphi + \cos^2\varphi = 1 \\ j \neq k: & a_{11}a_{12} + a_{21}a_{22} = 0 & \Rightarrow \cos\varphi\sin\varphi - \sin\varphi\cos\varphi = 0 \end{array}\right\} \text{OK!}$$

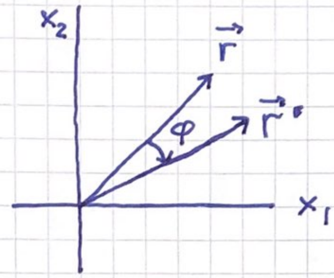
Expressed in 3D, rotation around z axis:

$$\mathbb{A} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Two interpretations of the transformation $\vec{r}\,' = \mathbb{A}\vec{r}$:



Passive



Active

Passive point of view:
Operator $\mathbb{A}$ rotates the coord. system, here counterclockwise. Vector $\vec{r}$ is fixed, $\vec{r}\,'$ expresses components of $\vec{r}$ in rotated coord. system.

Active point of view:
$\mathbb{A}$ rotates the vector $\vec{r}$, here clockwise, while the coord. system is fixed. $\vec{r}\,'$ contains components x_1' and x_2' after the rotation.

4.3 Formal properties of the transformation matrix

(see also MA1201, MA1202, TMA4115)

Notation: $\vec{r} \rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

- Two successive orthogonal transf. $\mathbb{B}$ followed by $\mathbb{A}$ equals a single orthog. transf. $\mathbb{C} = \mathbb{A}\mathbb{B}$:

$$x'_k = b_{kj}\, x_j \;;\; x''_i = a_{ik}\, x'_k = a_{ik}\, b_{kj}\, x_j \equiv C_{ij}\, x_j$$

$$\Rightarrow C_{ij} = a_{ik}\, b_{kj} \qquad \text{(remember summation convention)}$$

- In general, $\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}$ but $(\mathbb{A}\mathbb{B})\mathbb{C} = \mathbb{A}(\mathbb{B}\mathbb{C})$, i.e. orthog. transf. are associative but not commutative.

- The inverse transf. $\mathbb{A}^{-1}$, with matrix elements a^{-1}_{ij} (NB: $a^{-1}_{ij} \neq 1/a_{ij}$), brings $\vec{x}'$ back to $\vec{x}$:

$$x_i = a^{-1}_{ij}\, x'_j \quad \Rightarrow x'_k = a_{ki}\, x_i = a_{ki}\, a^{-1}_{ij}\, x'_j$$

$$\Rightarrow \underbrace{a_{ki}\, a^{-1}_{ij}}_{(\mathbb{A}\mathbb{A}^{-1})_{kj}} = \underbrace{\delta_{kj}}_{\mathbb{1}_{kj}} \quad \Rightarrow \mathbb{A}\mathbb{A}^{-1} = \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Also: $x_i = a^{-1}_{ij}\, x'_j = a^{-1}_{ij}\, a_{jk}\, x_k \Rightarrow a^{-1}_{ij}\, a_{jk} = \delta_{ik}$

$\Rightarrow \mathbb{A}^{-1}\mathbb{A} = \mathbb{1} \quad \Rightarrow \mathbb{A}$ and $\mathbb{A}^{-1}$ commute

- For an orthogonal matrix, $\mathbb{A}^{-1} = \tilde{\mathbb{A}}$ = transposed of $\mathbb{A}$

Proof: $a_{kl}\, a_{ki}\, a^{-1}_{ij} = \delta_{il}\, a^{-1}_{ij} = a^{-1}_{lj}$ (using orthog. cond.) and $a_{kl}\, a_{ki}\, a^{-1}_{ij} = a_{kl}\, \delta_{kj} = a_{jl}$ (using $\mathbb{A}\mathbb{A}^{-1} = \mathbb{1}$). Hence, $a^{-1}_{lj} = a_{jl} = \tilde{a}_{lj}$, i.e., $\mathbb{A}^{-1} = \tilde{\mathbb{A}}$

- Alternative expressions for orthog. conditions:

$$\tilde{\mathbb{A}}\mathbb{A} = \mathbb{1} \Rightarrow \tilde{a}_{ij} a_{jk} = \delta_{ik} \Rightarrow a_{ji} a_{jk} = \delta_{ik}$$

$$\mathbb{A}\tilde{\mathbb{A}} = \mathbb{1} \Rightarrow a_{ij} \tilde{a}_{jk} = \delta_{ik} \Rightarrow a_{ij} a_{kj} = \delta_{ik}$$

I.e., may sum over 1. or 2. index

- Determinants: $|\mathbb{A}\mathbb{B}| = |\mathbb{A}| \cdot |\mathbb{B}|$ (in general)
  $\Rightarrow |\tilde{\mathbb{A}}| \cdot |\mathbb{A}| = 1$ and since $|\tilde{\mathbb{A}}| = |\mathbb{A}|$ (— " —)
  we find $|\mathbb{A}|^2 = 1$, i.e., $|\mathbb{A}| = \pm 1$

- The orthog. transf. must be realizable for a rigid body, hence $|\mathbb{A}| = +1$

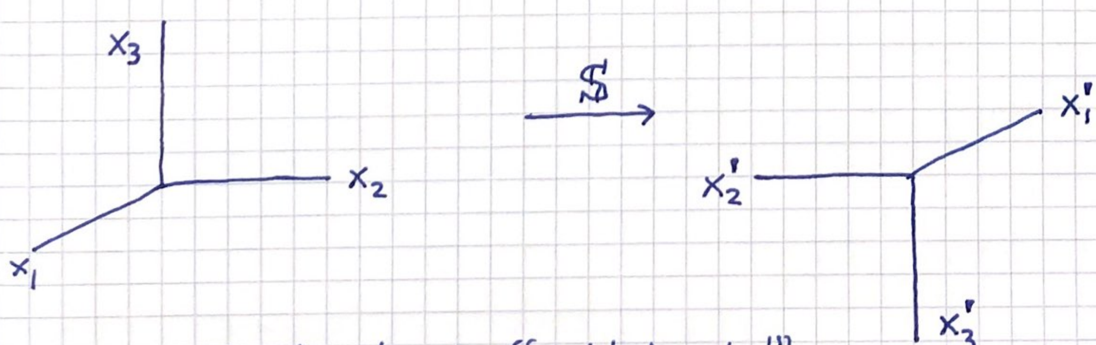
  Argument 1: $|\mathbb{A}|$ must evolve continuously from the unit matrix $\Rightarrow |\mathbb{A}| = |\mathbb{1}| = +1$.

  Argument 2, by an example:

  Consider the transf. $\vec{x}' = \mathbb{S}\vec{x}$ with

$$\mathbb{S} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow |\mathbb{S}| = -1$$

  i.e. reflection of all 3 axes, $x_1' = -x_1$ etc.



Not possible to change "right handed" rigid body into a "left handed" one by rotations and translation

4.4 The Euler angles

We need 3 independent coords. that specify the rigid body orientation, such that $|\mathbb{A}| = +1$. Common choice is the Euler angles, which are 3 successive rotation angles. $\mathbb{A} = \mathbb{B}\,\mathbb{C}\,\mathbb{D}$

1. Rot. $\varphi > 0$ around z axis, $xyz \to \xi\eta\zeta$

$$\boldsymbol{\xi} = \mathbb{D}\,\mathbb{x}, \quad \mathbb{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{\xi} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

$$\mathbb{D} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Rot. $\theta > 0$ around ξ axis, $\xi\eta\zeta \to \xi'\eta'\zeta'$

$$\boldsymbol{\xi}' = \mathbb{C}\,\boldsymbol{\xi}, \quad \boldsymbol{\xi}' = \begin{pmatrix} \xi' \\ \eta' \\ \zeta' \end{pmatrix}$$

$$\mathbb{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

3. Rot. $\psi > 0$ around ζ' axis, $\xi'\eta'\zeta' \to x'y'z'$

$$\mathbb{x}' = \mathbb{B}\,\boldsymbol{\xi}', \quad \mathbb{x}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\mathbb{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{x}' = \mathbb{A}\,\mathbb{x} = \mathbb{B}\,\mathbb{C}\,\mathbb{D}\,\mathbb{x}$$

$$\Rightarrow \quad a_{11} = \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi$$

$$a_{12} = \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi$$

$$a_{13} = \sin\psi\sin\theta$$

$$a_{21} = -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi$$

$$a_{22} = -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi$$

$$a_{23} = \cos\psi\sin\theta$$

$$a_{31} = \sin\theta\sin\varphi$$

$$a_{32} = -\sin\theta\cos\varphi$$

$$a_{33} = \cos\theta$$

Inverse transformation:

$$\mathbb{x} = \mathbb{A}^{-1}\mathbb{x}' = \tilde{\mathbb{A}}\,\mathbb{x}'$$

$$\tilde{a}_{ij} = a_{ji}$$

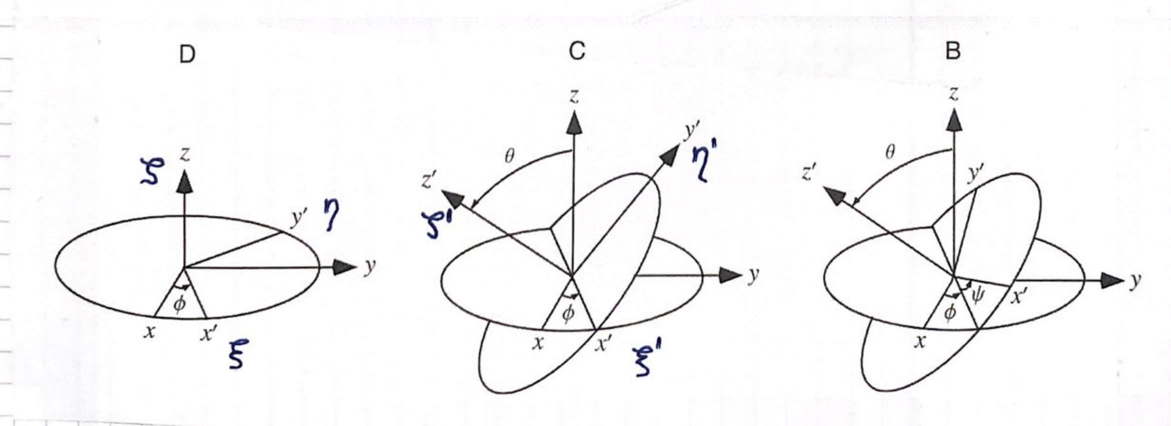


Fig: Wolfram MathWorld (E.W. Weisstein)

4.8 Infinitesimal rotations

In general, two finite rotations $\mathbb{A}$ and $\mathbb{B}$ do not commute.

Ex. 1 : $\mathbb{C}$ and $\mathbb{D}$ on p. 60.

Ex 2 : Matchbox rotated 90° around two orthogonal axes.

But infinitesimal transformations commute :

$$x_i' = x_i + \varepsilon_{ij} x_j = (\delta_{ij} + \varepsilon_{ij}) x_j \quad ; \quad \varepsilon_{ij} \ll 1$$

On matrix form : $\mathbb{X}' = (\mathbb{1} + \mathbb{E}) \mathbb{X}$

Two successive inf. transf. :

$$\left.\begin{aligned} (\mathbb{1} + \mathbb{E}_1)(\mathbb{1} + \mathbb{E}_2) &= \mathbb{1} + \mathbb{E}_1 + \mathbb{E}_2 \\ (\mathbb{1} + \mathbb{E}_2)(\mathbb{1} + \mathbb{E}_1) &= \mathbb{1} + \mathbb{E}_1 + \mathbb{E}_2 \end{aligned}\right\} \text{commutative!}$$

Inverse inf. transf. :

$$\mathbb{A}^{-1} = \mathbb{1} - \mathbb{E} \qquad \text{since} \quad \mathbb{A}\mathbb{A}^{-1} = (\mathbb{1} + \mathbb{E})(\mathbb{1} - \mathbb{E}) = \mathbb{1}$$

Orthogonality :

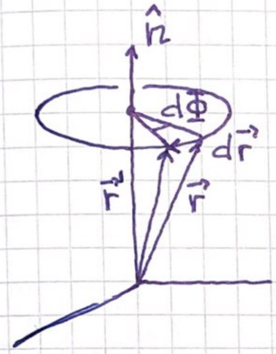
$$\tilde{\mathbb{A}} = \mathbb{1} + \tilde{\mathbb{E}} = \mathbb{A}^{-1} \Rightarrow \tilde{\mathbb{E}} = -\mathbb{E} \Rightarrow \tilde{\varepsilon}_{ij} = \varepsilon_{ji} = -\varepsilon_{ij}$$

i.e. $\mathbb{E}$ is antisymmetric 3×3 matrix on the form

$$\mathbb{E} = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbb{X}' - \mathbb{X} \equiv d\mathbb{X} = \mathbb{E}\mathbb{X} = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} dx_1 &= x_2\, d\Omega_3 - x_3\, d\Omega_2 \\ dx_2 &= x_3\, d\Omega_1 - x_1\, d\Omega_3 \\ dx_3 &= x_1\, d\Omega_2 - x_2\, d\Omega_1 \end{aligned} \qquad \Rightarrow d\vec{r} = \vec{r} \times d\vec{\Omega}$$



$d\vec{r} = \vec{r} \times d\vec{\Omega}$; $d\vec{\Omega} = \hat{n}\, d\Phi$

NB: $d\vec{\Omega}$ is <u>not</u> the differential of a finite vector $\vec{\Omega}$; $d\vec{\Omega}$ is a differential vector

$d\vec{r} = \vec{r}\,' - \vec{r}$ = change in the vector produced by an infinitesimal (clockwise) rotation of the vector

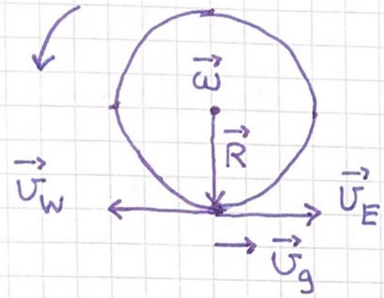
4.9 Rate of change of a vector

The change $d\vec{G}$ in an arbitrary (general) vector $\vec{G}$ is measured with different values in the coord. system fixed in the rigid body and the external coord. system:

$$(d\vec{G})_{body} \neq (d\vec{G})_{space}$$

The difference due to rotation of the body system in the space system is $\vec{G} \times d\vec{\Omega}$, and we fix the <u>sign</u> with an example.

A car drives east or west on the equator :



$\vec{\omega} = d\vec{\Omega}/dt$ out of the plane

$\vec{v}_g = \vec{\omega} \times \vec{R}$ = speed of ground measured in fixed external system

$\vec{v}_{E,W}$ = speed of car in rotating system

$$\Rightarrow \quad \vec{v}_c = \vec{v}_{E,W} + \vec{\omega} \times \vec{R}$$

= speed of car in external system

$$\Rightarrow \quad (d\vec{G})_{space} = (d\vec{G})_{body} + d\vec{\Omega} \times \vec{G}$$

$$\Rightarrow \quad \left(\frac{d\vec{G}}{dt}\right)_s = \left(\frac{d\vec{G}}{dt}\right)_b + \vec{\omega} \times \vec{G}$$

$\vec{\omega} = d\vec{\Omega}/dt$ = instantaneous angular velocity

Operator relation:

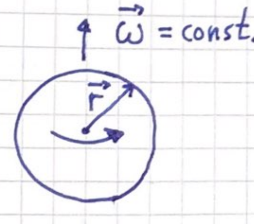
$$\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_b + \vec{\omega} \times$$

Ex: $\vec{G} = \vec{r} \quad \Rightarrow \quad \vec{v}_s = \vec{v}_b + \vec{\omega} \times \vec{r}$ (as above)

(For a more formal derivation of the operator relation, see Goldstein or handwritten compendium.)

4.10 Coriolis force and centrifugal force

s = approximate inertial system fixed relative to some nearby stars



b = system rotating with the earth

Assume common axes at a given instant.

$$\vec{G} = \vec{r} \quad \Rightarrow \quad \vec{v}_s = \vec{v}_b + \vec{\omega} \times \vec{r}$$

$$\vec{G} = \vec{v}_s \quad \Rightarrow \quad \left(\frac{d\vec{v}_s}{dt}\right)_s = \left(\frac{d\vec{v}_s}{dt}\right)_b + \vec{\omega} \times \vec{v}_s$$

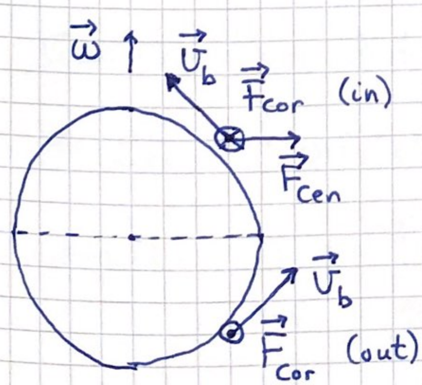
$$\left(\frac{d\vec{v}_s}{dt}\right)_b = \left(\frac{d\vec{v}_b}{dt}\right)_b + \left[\frac{d}{dt}(\vec{\omega} \times \vec{r})\right]_b = \vec{a}_b + \vec{\omega} \times \vec{v}_b$$

$$\vec{\omega} \times \vec{v}_s = \vec{\omega} \times \vec{v}_b + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\Rightarrow \vec{a}_s = \vec{a}_b + 2\,\vec{\omega} \times \vec{v}_b + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

N2 in the inertial frame s, $\vec{F} = m\vec{a}_s$, can now be expressed as $\vec{F}_b = m\vec{a}_b$ with

$$\vec{F}_b = \vec{F} + \underbrace{2m\,\vec{v}_b \times \vec{\omega}}_{\text{Coriolis force}} \underbrace{- m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{Centrifugal force}}$$



$\vec{F}_{cor}$ ⇒ right deflection on northern hemisphere; left deflection on southern hemisphere

Affects wind direction: