7. Special theory of relativity

7.1 Introduction

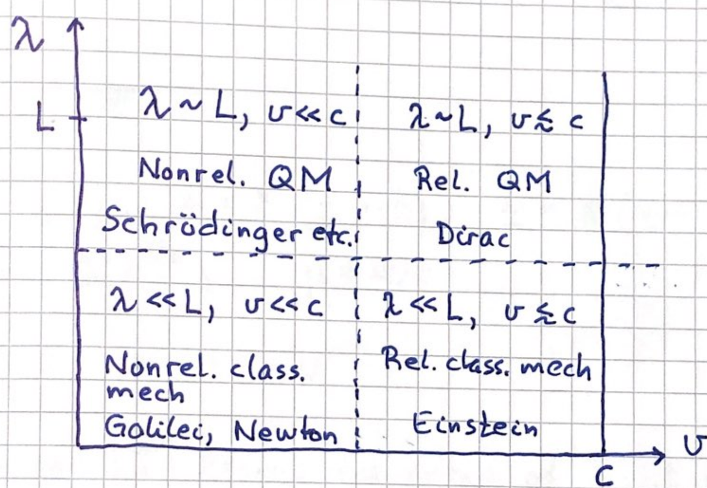
L = linear size of system

$\lambda = h/p$ = de Broglie wavelength for particle (body)

(p = momentum, h = Planck constant)

v = typical particle velocity

c = speed of light



We will consider systems with $\lambda \ll L$ and $v \lesssim c$, described by Einstein's special theory of relativity (SR)
[See GPS 7.11 for a very brief outline of the general theory (GR)]

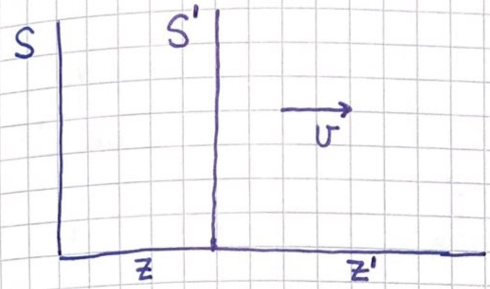
Galilei: Assume two inertial systems S and S'; S' moves with velocity $\vec{v}$ relative to S. A particle's coordinates $\vec{r}$ and $\vec{r}\,'$ and times t and t' observed in S and S' are related as $\vec{r}\,' = \vec{r} - \vec{v} \cdot t$ and $t' = t$. (GT)
The speed of light is different in S and S', $c' = c - v$.

Experiments: $c' = c$ [Michelson - Morley etc]

Einstein's two postulates:

1. The laws of physics are the same in all inertial systems
2. The speed of light is ———— " ————

7.2 The Lorentz transformation (LT)



Assume common origin at $t = t' = 0$ and $\vec{v} = v\hat{z}$

LT: $x' = x, \quad y' = y, \quad z' = \gamma(z - vt), \quad t' = \gamma\left(t - \frac{v}{c^2} z\right)$

$$\gamma = (1 - \beta^2)^{-1/2}, \qquad \beta = v/c$$

Inverse LT: $v \to -v$

Will show that LT leaves the speed of light invariant.
Consider source of light wave in origo of S' at $t' = 0$.
Equation for the light front:

$$r' = ct' \Rightarrow (r')^2 = (ct')^2 \Rightarrow (x')^2 + (y')^2 + (z')^2 = c^2 (t')^2$$

Use LT to find eqn. for light front in S:

$$(x')^2 + (y')^2 + (z')^2 - c^2 (t')^2$$
$$= x^2 + y^2 + \gamma^2 (z - vt)^2 - c^2 \gamma^2 \left(t - \frac{v}{c^2} z\right)^2$$
$$= \text{.... some algebra ....} = x^2 + y^2 + z^2 - c^2 t^2$$

$\Rightarrow$ Light front has speed c in both S' and S

If $v \ll c$: $x' = x, \; y' = y, \; z' = z - vt, \; t' = t$; $\quad LT \xrightarrow{v \ll c} GT$

Inverse transf.: $x = x', \; y = y', \; z = \gamma(z' + vt'), \; t = \gamma\left(t' + \frac{v}{c^2} z'\right)$

[Obvious: S moves with speed $-v$ relative to S']

LT with vectors ($\vec{v} = c\vec{\beta}$) :

$$\vec{r}' = \vec{r} + (\gamma - 1)\frac{(\vec{\beta}\cdot\vec{r})\,\vec{\beta}}{\beta^2} - \gamma c t \vec{\beta}$$

$$t' = \gamma t - \frac{\gamma}{c}\vec{\beta}\cdot\vec{r}$$

LT is linear $\Rightarrow$ may use ideas/results from chapter 4

Minkowski space :

4D space with axes $x_1 = x, x_2 = y, x_3 = z$ and $x_4 = ict$

$$\Rightarrow x^2 + y^2 + z^2 - c^2t^2 = \sum_{\mu=1}^{4} x_\mu x_\mu \equiv x_\mu x_\mu \quad \text{[Sum conv.]}$$

Index notation : Greek for 4-vectors ($\alpha, \beta, \mu \ldots$)
Roman for 3-vectors ($i, j, k \ldots$)

$\Rightarrow$ Invariance of the speed of light can be expressed as

" $x_\mu x_\mu$ is an invariant "

Or: The norm (abs. value) of the "position vector" in Minkowski space is constant under a LT

From 4.2 (p.55) : A linear transf. that leaves the norm of the vector unchanged is an orthogonal transf.

$\Rightarrow$ LT is an orthogonal transf. in Minkowski space

LT on matrix form:

$$\mathbb{x}' = \mathbb{L}\,\mathbb{x} \quad \text{or} \quad x'_\mu = L_{\mu\nu} x_\nu$$

With relative velocity along x_3 axis ($\vec{v} = v\hat{x}_3$) :

$$\mathbb{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}$$

From 4.3 (p. 58) : $\mathbb{L}^{-1} = \tilde{\mathbb{L}}$ (= transposed of $\mathbb{L}$)

Inverse LT : $\mathbb{X} = \mathbb{L}^{-1}\mathbb{X}' = \tilde{\mathbb{L}}\,\mathbb{X}' \Rightarrow x_\mu = L_{\nu\mu} x'_\nu$

$$\mathbb{L}^{-1} = \tilde{\mathbb{L}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{pmatrix}$$

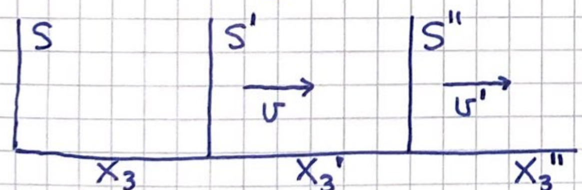
[$v \to -v$ as seen before]

Trigonometric convenience :

From example p. 56 : $\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ repr. rotation in a plane

Then, with $\gamma = \cos\varphi$ and $i\beta\gamma = \sin\varphi$, a LT can be viewed as a rotation an <u>imaginary</u> angle φ in the $x_3 x_4$ plane.

Ex : Velocity addition



v = speed of S' rel. to S

v' = — " — S'' — " — S'

v'' = — " — S'' — " — S

v'' = ? GT : $v'' = v' + v$

$$\mathbb{X}'' = \mathbb{L}'\mathbb{X}' = \mathbb{L}'\mathbb{L}\,\mathbb{X} = \mathbb{L}''\mathbb{X}$$

$$\Rightarrow L''_{33} = \cos\varphi'\cos\varphi - \sin\varphi'\sin\varphi = \cos(\varphi'+\varphi) = \cos\varphi''$$

$$L''_{34} = \cos\varphi'\sin\varphi + \sin\varphi'\cos\varphi = \sin(\varphi'+\varphi) = \sin\varphi''$$

$$\Rightarrow \tan\varphi'' = \frac{L''_{34}}{L''_{33}} = \dots = \frac{\tan\varphi + \tan\varphi'}{1 - \tan\varphi\tan\varphi'}$$

$$\Rightarrow i\beta'' = \frac{i\beta + i\beta'}{1 + \beta\beta'} \Rightarrow \underline{v'' = \frac{v + v'}{1 + vv'/c^2}}$$; Einstein's addition formula

7.3 Covariant four-dimensional formulation

We will address Einstein's 1. postulate (the principle of relativity):

Are the laws of physics the same in all inertial systems?

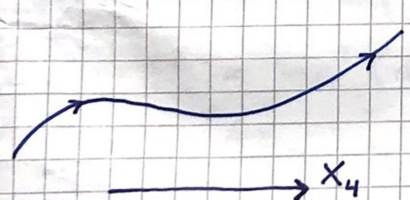
Or:

Are the equations (describing the laws of physics) covariant, i.e., invariant in form, under a LT?

Laws-of-physics-equations provide the relation between scalars, vectors and tensors (expressed by matrices). Scalars and vectors are special cases of tensors. Hence, we must in general have <u>covariance of tensor equations</u>:

If $C_{\mu\nu} = D_{\mu\nu}$, then $C'_{\mu\nu} = D'_{\mu\nu}$ after a LT

Consider a particle (or observer), moving in 3D space as time goes by; $u^2 = (dx_i/dt)\cdot(dx_i/dt)$:



The particle's world line (or: proper line) in Minkowski space (or: spacetime)

dx_μ = small element along the world line

Proper time τ : $dx_\mu dx_\mu \equiv -c^2 d\tau^2$ (def. of τ)

Let us show that

$$dx'_\mu dx'_\mu = dx_\mu dx_\mu$$

i.e., the same in S' and S, i.e., a Lorentz invariant

From 7.2 (and 4.3):

$$\mathbb{X}' = \mathbb{L}\,\mathbb{X} \; ; \quad x'_\mu = L_{\mu\nu} x_\nu$$

$$\mathbb{L}^{-1} = \tilde{\mathbb{L}} \quad \Rightarrow \mathbb{L}\tilde{\mathbb{L}} = \tilde{\mathbb{L}}\,\mathbb{L} = \mathbb{1} \quad (\text{since } \mathbb{L}\mathbb{L}^{-1} = \mathbb{L}^{-1}\mathbb{L} = \mathbb{1})$$

Then:

$$\begin{aligned} dx'_\mu\, dx'_\mu &= L_{\mu\nu}\, dx_\nu \cdot L_{\mu\alpha}\, dx_\alpha \\ &= L_{\mu\nu}\, \tilde{L}_{\alpha\mu}\, dx_\nu\, dx_\alpha \\ &= \delta_{\alpha\nu}\, dx_\nu\, dx_\alpha = dx_\alpha\, dx_\alpha = dx_\mu\, dx_\mu \end{aligned}$$

I.e., $dx'_\mu\, dx'_\mu = dx_\mu\, dx_\mu = -c^2 d\tau^2$

Time dilation:

Assume particle at rest in S'; $u = v$ = speed of S' rel. to S

$\Rightarrow dx'_\mu = (0, 0, 0, ic\,dt')$

$\Rightarrow dx'_\mu\, dx'_\mu = -c^2 dt'^2 = -c^2 d\tau^2$

$\Rightarrow d\tau = dt'$ = time measured on clock at rest

Measured in S: $dx_\mu = (dx, dy, dz, ic\,dt)$

$\Rightarrow dx_\mu\, dx_\mu = dx^2 + dy^2 + dz^2 - c^2 dt^2 = -c^2 d\tau^2$

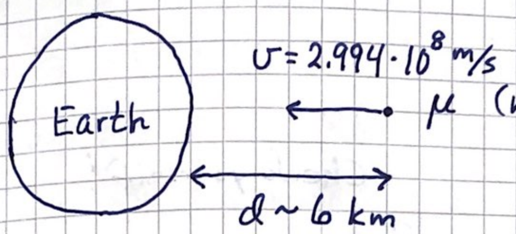
$\Rightarrow u^2 - c^2 = -c^2 (d\tau/dt)^2$

$\Rightarrow dt = \dfrac{d\tau}{\sqrt{1-\beta^2}} > d\tau \qquad (\beta = u/c = v/c)$

$\Rightarrow dt = \gamma\, d\tau$ = time measured on a clock in S, where the particle moves with speed v, is longer than the time measured on a clock in S', where the particle is at rest

Or: Moving clocks run slow.

Ex: Lifetime of muons in the atmosphere



μ (muon); $\tau = 2\ \mu s$
= proper lifetime

Nonrel.: $\ell = v \cdot \tau \approx 600\text{m} \ll d$

Rel.: $t = \gamma \tau \approx 32\ \mu s$ = lifetime measured on earth
$\ell = v \cdot t \approx 9.6$ km $> d$

$\Rightarrow$ Muons created in the atmosphere arrive on earth during their lifetime, in agreement with experiment.
[Bailey et al, Physics Letters 55 B, 420 (1975)]

Classification of 4-vectors:

space-like : $x_\mu x_\mu > 0$
time-like : $x_\mu x_\mu < 0$
light-like : $x_\mu x_\mu = 0$

Assume S' has speed v along $\hat{z}$ relative to S. Consider events 1 and 2 described by $x_{1\mu}$ and $x_{2\mu}$ in S. The difference $X_\mu = x_{1\mu} - x_{2\mu}$ has norm

$$X_\mu X_\mu = (z_1 - z_2)^2 - c^2 (t_1 - t_2)^2$$

The time difference in S' is

$$t_1' - t_2' = \gamma \left[t_1 - t_2 - \frac{v}{c^2}(z_1 - z_2) \right] = \frac{\gamma}{c} \left[c(t_1 - t_2) - \frac{v}{c}(z_1 - z_2) \right]$$

space-like $X_\mu \Rightarrow X_\mu X_\mu > 0 \Rightarrow c < \left| \frac{z_1 - z_2}{t_1 - t_2} \right|$

$\Rightarrow$ 1 and 2 cannot be connected with a signal with speed c

$\Rightarrow$ the two events cannot influence each other

However, we can find a $v \leq c$ that gives

$$t_1' - t_2' = 0 \quad \text{or} \quad t_1' - t_2' \gtrless 0$$

I.e., before and after are not uniquely determined.

time-like $X_\mu \Rightarrow X_\mu X_\mu < 0 \Rightarrow c > \left| \frac{z_1 - z_2}{t_1 - t_2} \right|$

$\Rightarrow$ 1 and 2 can be connected with a signal with speed c

$\Rightarrow$ the two events can influence each other

Now, $t_1' - t_2'$ and $t_1 - t_2$ have the same sign

I.e., before and after are uniquely determined

Causality : Cause before effect.

4-velocity: $u_\mu \equiv dx_\mu / d\tau = (dx_i, ic\,dt)/d\tau$

$$\Rightarrow u_i = dx_i / d\tau = \gamma\, dx_i / dt = \gamma v_i$$

$$u_4 = dx_4 / d\tau = ic\, dt/d\tau = ic\gamma$$

$$\Rightarrow u_\mu = \gamma(\vec{v}, ic)$$

Time-like :

$$u_\mu u_\mu = \gamma^2 v^2 - \gamma^2 c^2 = \frac{v^2 - c^2}{1 - v^2/c^2} = -c^2 < 0$$

4-current density :

The continuity equation is $\nabla \cdot \vec{j} + \partial \rho / \partial t = 0$

with $\vec{j}$ = current density, ρ = charge density

With $j_\mu = (\vec{j}, ic\rho)$: $\frac{\partial}{\partial x_\mu} j_\mu = 0$ or $\partial_\mu j_\mu = 0$

Due to length contraction, the charge density is larger for moving charges than for charges at rest:

$$\rho = \gamma \rho_0 > \rho_0$$

Hence,

$$j_\mu = (\vec{j}, ic\rho) = (\rho\vec{v}, ic\rho) = (\gamma\rho_0\vec{v}, ic\gamma\rho_0) = \rho_0 u_\mu$$

Let us show that the continuity eqn. is covariant. We must show that $\partial/\partial x_\mu$ transforms like dx_μ under a LT. We have (from p. 83) $dx_\mu = \underbrace{L_{\nu\mu}}_{= \partial x_\mu/\partial x'_\nu} dx'_\nu$

$$\Rightarrow \frac{\partial}{\partial x'_\nu} = \frac{\partial x_\mu}{\partial x'_\nu}\frac{\partial}{\partial x_\mu} = L_{\nu\mu}\frac{\partial}{\partial x_\mu}$$

$\Rightarrow$ Both $\partial/\partial x_\mu$ and $j_\mu = \rho_0 u_\mu$ transform as dx_μ under a LT

$\Rightarrow$ $\partial_\mu j_\mu$ is a Lorentz invariant scalar

$\Rightarrow$ The cont. eqn. <u>is</u> covariant

Maxwell's equations and the 4-potential:

From 1.5; assuming vacuum $(\vec{P} = \vec{M} = 0)$:

$$\vec{B} = \nabla \times \vec{A} \quad ; \quad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

(using $\nabla\cdot\vec{B} = 0$ and $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$)

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \Rightarrow \nabla^2\phi + \frac{\partial}{\partial t}\nabla\cdot\vec{A} = -\rho/\varepsilon_0$$

$$\nabla\times\vec{B} = \mu_0\vec{j} + \varepsilon_0\mu_0\,\partial\vec{E}/\partial t \quad ; \quad \varepsilon_0\mu_0 = 1/c^2$$

$$\Rightarrow \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla\left(\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\mu_0\vec{j}$$

(using the identity $\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$; see Assignment 2, Question 3)

Maxwell's 4 eqns. for $\vec{E}$ and $\vec{B}$ are reduced to 2 coupled eqns. for ϕ and $\vec{A}$; these may be decoupled since we have some <u>freedom</u> in our choice for ϕ and $\vec{A}$:

$\vec{B} = \nabla \times \vec{A} \quad \Rightarrow \quad \vec{A} \to \vec{A} + \nabla \chi$ leaves $\vec{B}$ unchanged for arbitrary scalar χ, since $\nabla \times \nabla \chi = 0$

Since $\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t$, we must also let $\phi \to \phi - \partial\chi/\partial t$, to leave $\vec{E}$ unchanged.

This freedom to adjust ϕ and $\vec{A}$ is called <u>gauge invariance</u>.

Lorenz gauge (Ludvig Lorenz, Danish physicist, 1829-1891):

Choose a transformation that obeys

$$\nabla^2 \chi - \frac{1}{c^2}\partial^2\chi/\partial t^2 = 0$$

Then, the Lorenz condition

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\partial\phi/\partial t = 0$$

is gauge invariant,

$$\nabla \cdot (\vec{A} + \nabla\chi) + \frac{1}{c^2}\frac{\partial}{\partial t}(\phi - \partial\chi/\partial t)$$

$$= \underbrace{\nabla \cdot \vec{A} + \frac{1}{c^2}\partial\phi/\partial t}_{=0} + \underbrace{\nabla^2\chi - \frac{1}{c^2}\partial^2\chi/\partial t^2}_{=0} = 0$$

The Lorenz condition decouples the eqns. for $\vec{A}$ and ϕ:

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = -\mu_0\vec{j}$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\rho/\varepsilon_0$$

The 4-potential: $A_\mu = (\vec{A}, i\phi/c)$

$\Rightarrow$ Lorenz condition: $\partial_\mu A_\mu = 0$ $\quad$ [since $\frac{\partial}{\partial x_4} = \frac{\partial}{ic\,\partial t}$]

Eqn. for A_μ: $\Box^2 A_\mu = -\mu_0 j_\mu$; $\quad \Box^2 \equiv \partial_\mu\partial_\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

= D'Alembert's operator

$\Rightarrow$ A_μ transforms like $j_\mu = (\vec{j}, ic\rho)$ and u_μ and x_μ under a Lorentz transf. (Hendrik Lorentz, Dutch physicist, 1853-1928)

$\Rightarrow$ Maxwell's eqns. are covariant, i.e., consistent with SR !

7.4 Relativistic dynamics

N2, $F = ma$, is invariant under a GT, not under a LT

$\Rightarrow$ We must generalize Newtonian dynamics such that it

(a) becomes covariant, and

(b) reduces to $\frac{d}{dt}(m v_i) = F_i$ when $v_i \ll c$

Natural attempt:

$$\frac{d}{d\tau}(m u_\mu) = K_\mu \qquad \text{with} \quad \begin{cases} m = \text{invariant mass} \\ \tau = \text{proper time} \\ u_\mu = \text{4-velocity} \\ K_\mu = \text{Minkowski force} \end{cases}$$

Since $d\tau = dt/\gamma \to dt$ and $u_i = \gamma v_i \to v_i$ when $v \to 0$,

we must have $K_i \to F_i$ when $v \to 0$ (i.e. $v_i \ll c$)

Let us consider Maxwell's EM theory, since it is covariant, with a known force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$.

The Lorentz force in terms of $\vec{A}$ and ϕ (i.e. A_μ):

$$\vec{F} = q\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \nabla \times \vec{A}\right)$$

$$\overset{p.17}{\Longrightarrow} F_i = -q\left[\partial_i(\phi - \vec{v}\cdot\vec{A}) + \frac{dA_i}{dt}\right]$$

Since $u_\mu A_\mu = \gamma v_i A_i + \gamma i c \frac{i\phi}{c} = \gamma(\vec{v}\cdot\vec{A} - \phi)$

and $dt = \gamma\, d\tau$, we have

$$F_i = -q\left[-\frac{1}{\gamma}\partial_i(u_\mu A_\mu) + \frac{1}{\gamma}\frac{dA_i}{d\tau}\right]$$

$$= \frac{q}{\gamma}\left[\partial_i(u_\mu A_\mu) - \frac{dA_i}{d\tau}\right]$$

When $v \ll c$, $\gamma \to 1$, and we identify

$$K_i = q\left[\partial_i(u_\mu A_\mu) - \frac{dA_i}{d\tau}\right] = \gamma F_i$$

and $$K_\mu = q\left[\partial_\mu(u_\nu A_\nu) - \frac{dA_\mu}{d\tau}\right]$$

4-momentum: $p_\mu = m u_\mu = (m\gamma \vec{v}, ic\gamma m)$

$$\Rightarrow K_\mu = \frac{dp_\mu}{d\tau}$$

$$\Rightarrow K_i = \gamma F_i = \frac{dp_i}{dt/\gamma} \Rightarrow \vec{F} = \frac{d\vec{p}}{dt}$$ ☺

Energy and power:

$$u_\mu K_\mu = u_\mu \frac{d}{d\tau}(m u_\mu) = \frac{d}{d\tau}\left(\tfrac{1}{2} m u_\mu u_\mu\right) = 0$$

since $u_\mu u_\mu = -c^2$ (see p. 87)

$$\Rightarrow u_i K_i + u_4 K_4 = 0$$

$$\Rightarrow \gamma v_i \cdot \gamma F_i + ic\gamma \cdot K_4 = 0$$

$$\Rightarrow K_4 = \frac{i\gamma}{c} \vec{F}\cdot\vec{v} \qquad \Rightarrow K_\mu = \gamma\left(\vec{F}, \frac{i}{c}\vec{F}\cdot\vec{v}\right)$$

We combine $dp_4/d\tau = K_4$ with $p_4 = ic\gamma m$ and $K_4 = (i\gamma/c)\,\vec{F}\cdot\vec{v}$ and find

$$\vec{F}\cdot\vec{v} = \frac{d}{dt}(\gamma m c^2)$$

Since $\vec{F}\cdot\vec{v}$ is added power, we identify the total energy $E = \gamma m c^2$, and $p_4 = i\gamma m c = iE/c$

In summary: $\boxed{\vec{F} = d\vec{p}/dt, \quad \vec{F}\cdot\vec{v} = dE/dt, \quad p_\mu = (\vec{p}, iE/c)}$

- Rest energy $(v=0)$: $E_0 = mc^2$
- Kinetic energy: $T = E - E_0 = (\gamma - 1)mc^2$
- Low speed: $\gamma \approx 1 + \frac{1}{2}\beta^2 \Rightarrow T \approx \frac{1}{2}mv^2$
- If the particle is in an external field, the potential energy V must be added to $\gamma m c^2$

Transformation of momentum and energy:

Since $p_\mu = m\, dx_\mu / d\tau$, p_μ will transform as x_μ,

$$\mathbb{P}' = \mathbb{L}\,\mathbb{P}$$

$\Rightarrow$ $p_1' = p_1$, $p_2' = p_2$, $p_3' = \gamma p_3 + i\beta\gamma p_4 = \gamma\left(p_3 - \frac{v}{c^2}E\right)$

$p_4' = -i\beta\gamma p_3 + \gamma p_4 = -i\beta\gamma p_3 + \gamma \frac{iE}{c}$ and $p_4' = \frac{iE'}{c}$

$$\Rightarrow \quad E' = \gamma\,(E - v p_3)$$

The invariant $p_\mu p_\mu$ in the particle's rest frame ($\vec{p}=0$, $\gamma=1$):

$$p_\mu p_\mu = -E_0^2/c^2 = -m^2c^2$$

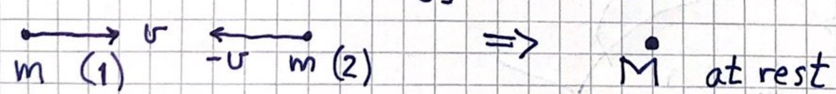
Hence, in general: $p^2 - E^2/c^2 = -m^2c^2$

$$\Rightarrow \quad \boxed{E^2 = p^2c^2 + m^2c^4}$$

Ex 1: Photons; $m=0 \Rightarrow E = pc$. Planck: $E = h\nu$

$\Rightarrow pc = p\lambda\nu = h\nu \Rightarrow p = h/\lambda$ (de Broglie)

Ex 2: Fusion; energy conservation and mass change



No external force $\Rightarrow$ conservation of $\vec{p} = \vec{p}_1 + \vec{p}_2$ $(=0)$

LT mixes E and $\vec{p}$ $\Rightarrow$ energy also conserved

$\Rightarrow$ Total 4-momentum $P_\mu = (\vec{p}, iE/c)$ conserved

Conservation of P_4: $\gamma mc^2 + \gamma mc^2 = Mc^2$

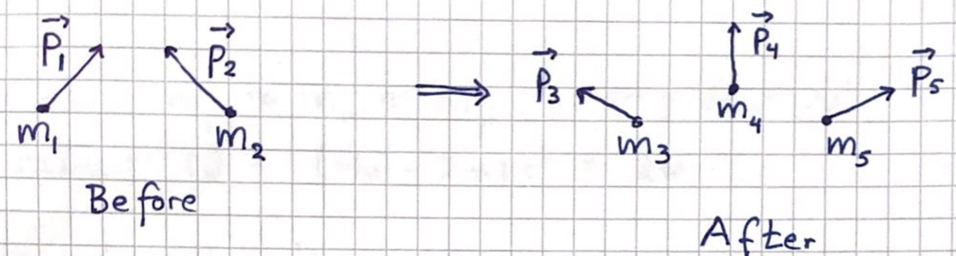
$$\Rightarrow M = 2\gamma m > 2m$$

Kin. energy $\rightarrow$ Rest energy (mass): $\Delta M = M - 2m = 2m(\gamma - 1)$

7.5 Relativistic kinematics

Relevance: Particle physics, High energy physics

Typical situation: 2 particles collide and form 2 or more new particles. Details of the interaction unknown, but conservation of momentum and energy, i.e., cons. of 4-momentum applies to "before" and "after".



$$P_\mu = p_{1\mu} + p_{2\mu} \qquad P'_\mu = p_{3\mu} + p_{4\mu} + p_{5\mu} \quad (+\ldots)$$

For each particle: $p_\mu = (\vec{p}, iE/c)$

Convenient inertial system:

The center-of-momentum system (COM) where
$\vec{P} = \sum_j \vec{p}_j = 0$

In the COM system we have

$$P_\mu P_\mu = P'_\mu P'_\mu = -M^2 c^2$$

which defines the equivalent mass M of the system.

Threshold energy = Minimum energy required for a reaction to take place; corresponds to all reaction products at rest in the COM system after the collision. In that case, $M = \sum_r m_r$ = sum of mass of rxn. products

The Q-value of a reaction:

$Q = \left[\sum_r m_r - (m_1 + m_2)\right] c^2$ = rest energy created in the reaction

Ex: Antiproton production

$$p + n \rightarrow p + n + p + \bar{p} \qquad [\text{or: } p+p \rightarrow p+p+p+\bar{p}\,]$$

P.A.M. Dirac, Nobel Lecture 12.12.1933 : Prediction of $\bar{p}$

O. Chamberlain, E. Segrè, C. Wiegand, T. Ypsilantis, Phys. Rev. 100, 947 (1955) : Observation of $\bar{p}$

[Chamberlain and Segrè, Nobel Prize 1959]

Masses : $m_p \approx m_n \approx m_{\bar{p}} \approx 938\ \mathrm{MeV}/c^2 = m$

Q-value : $Q = (4m - 2m)\,c^2 = 2mc^2$

Calculate the threshold energy

(a) T_p with n at rest initially

(b) $T_p = T_n$ for head-on collision with $\vec{p}_p = -\vec{p}_n$

$\hookrightarrow$ $\boxed{\text{Assignment 11}}$

(a) We consider the covariant $P_\mu P_\mu$ in the lab system, to get T_p into the equation, and compare it with $P'_\mu P'_\mu$ after the collision in the COM system, where $M = 4m$. In other words,

$$P_\mu P_\mu = -M^2c^2 = -16m^2c^2$$

The left hand side is

$$P_\mu P_\mu = (p_{p\mu} + p_{n\mu})\cdot(p_{p\mu} + p_{n\mu})$$

$$= p_{p\mu}\,p_{p\mu} + p_{n\mu}\,p_{n\mu} + 2\,p_{p\mu}\,p_{n\mu}$$

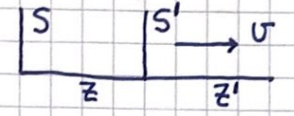
$$= -m^2c^2 - m^2c^2 + 2\Big(\underbrace{\vec{p}_p\cdot\vec{p}_n}_{=0} + \frac{iE_p}{c}\cdot\frac{iE_n}{c}\Big)$$

Here, $E_n = mc^2$ and $E_p = mc^2 + T_p$

$$\Rightarrow -4m^2c^2 - 2mT_p = -16m^2c^2 \quad \Rightarrow T_p = 6mc^2 = \underline{\underline{5.57\ \mathrm{GeV}}}$$

7.6 The electromagnetic field tensor and transformation of $\vec{E}$ and $\vec{B}$

As usual, we assume:



Definition of EM field tensor:

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

i.e. an antisymmetric tensor, $F_{\nu\mu} = -F_{\mu\nu}$

We have: $A_\mu = (\vec{A}, i\phi/c)$; $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi$; $\vec{B} = \nabla \times \vec{A}$

Hence:

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1/c \\ -B_3 & 0 & B_1 & -iE_2/c \\ B_2 & -B_1 & 0 & -iE_3/c \\ iE_1/c & iE_2/c & iE_3/c & 0 \end{pmatrix}$$

Lorentz transformation of rank 2 tensor:

$$F'_{\mu\nu} = L_{\mu\alpha} L_{\nu\beta} F_{\alpha\beta}$$

with

$$L_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}$$

Now, it is straightforward to find $\vec{E}'$ and $\vec{B}'$ expressed in terms of $\vec{E}$, $\vec{B}$ and $\vec{v}$.

Let us derive a couple of components, then write down the remaining components, and finally consider the nonrelativistic limit $v \ll c$.

$\mu = 4,\ \nu = 1:$

$$\frac{i}{c}E_1' = F_{41}' = L_{4\alpha}L_{1\beta}F_{\alpha\beta} = -i\beta\gamma B_2 + \gamma\cdot\frac{i}{c}E_1$$

$$\Rightarrow E_1' = \gamma(E_1 - vB_2)$$

Assignment 11 : $E_2' = \gamma(E_2 + vB_1)$; $E_3' = E_3$

If $v \ll c$, $\gamma = 1$, and $\vec{E}' = \vec{E} + \vec{v}\times\vec{B}$

$i = 2,\ k = 3:$

$$B_1' = F_{23}' = L_{2\alpha}L_{3\beta}F_{\alpha\beta} = L_{33}F_{23} + L_{34}F_{24}$$

$$= \gamma B_1 + i\beta\gamma\cdot\left(-\frac{i}{c}E_2\right) = \gamma\left(B_1 + \frac{\beta}{c}E_2\right)$$

Assignment 11 : $B_2' = \gamma\left(B_2 - \frac{\beta}{c}E_1\right)$; $B_3' = B_3$

If $v \ll c$, $\gamma = 1$, and $\vec{B}' = \vec{B} - \frac{1}{c^2}(\vec{v}\times\vec{E})$