CLASSICAL MECHANICS

1. Fundamental principles

1.1 Single particle

Position : $\vec{r}$

Velocity : $\vec{v} = \frac{d\vec{r}}{dt} \quad (= \dot{\vec{r}})$

Acceleration : $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \quad (= \dot{\vec{v}} = \ddot{\vec{r}})$

Linear momentum : $\vec{p} = m\vec{v}$; m = particle mass

Total force on particle : $\vec{F}$

Newton's 2. law (N2), valid in inertial frame: $\vec{F} = \frac{d\vec{p}}{dt}$

with constant mass m : $\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$

Conservation law for linear momentum:

$$\vec{F} = 0 \quad \Rightarrow \quad \vec{p} \text{ is conserved}$$

Angular momentum, relative to the origin:

$$\vec{L} = \vec{r} \times \vec{p}$$

Torque, relative to the origin:

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad (\text{GPS} : \vec{N})$$

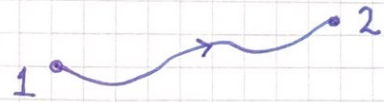
$$\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d}{dt}\underbrace{(\vec{r} \times \vec{p})}_{=\vec{L}} - \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{=0} \Rightarrow \vec{\tau} = \frac{d\vec{L}}{dt}$$

Conservation law for angular momentum:

$$\vec{\tau} = 0 \quad \Rightarrow \quad \vec{L} \text{ is conserved}$$

Work done by force $\vec{F}$ when particle moves from 1 to 2:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s}$$



With constant m (assumed if otherwise not stated):

$$\vec{F} \cdot d\vec{s} = m \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt = m \cdot \frac{1}{2}\, d v^2$$

$$\Rightarrow W_{12} = \frac{1}{2} m \left(v_2^2 - v_1^2 \right) = T_2 - T_1$$

Kinetic energy of particle: $T = \frac{1}{2} m v^2$

Conservative system if W_{12} is independent of path from 1 to 2; then $\oint \vec{F} \cdot d\vec{s} = 0$, and we may write

$$\vec{F} = -\nabla V(\vec{r}) \quad ; \quad V = \text{potential energy (or simply potential)}$$

Since $\nabla V = \nabla (V + \text{constant})$, the choice of $V = 0$ is arbitrary.

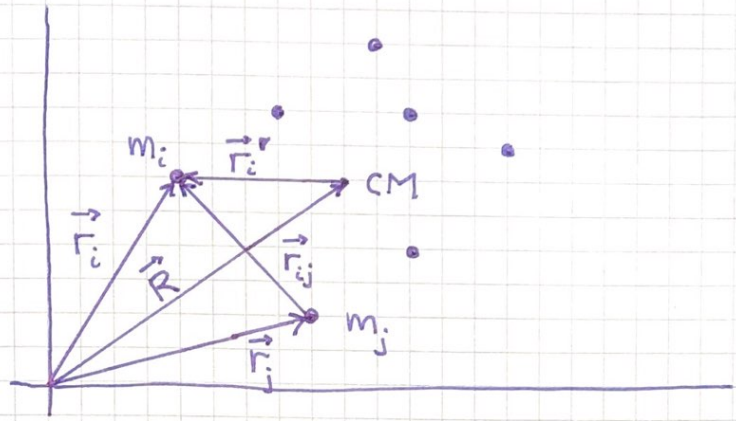
For cons. system:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s} = \int_2^1 \nabla V \cdot d\vec{s} = \int_2^1 dV = V_1 - V_2$$

$$\Rightarrow T_2 - T_1 = V_1 - V_2 \Rightarrow T_1 + V_1 = T_2 + V_2$$

Conservation law for particle energy:
In a conservative system, the total energy $T + V$ is conserved

1.2 System of particles



N2 for m_i : $\sum_{j \neq i} \vec{F}_{ji} + \vec{F}_i^{(e)} = \dot{\vec{p}}_i$

internal force from j on i — external force on i

Assume $\vec{F}_{ij}$ obeys N3, i.e., $\vec{F}_{ij} = -\vec{F}_{ji}$ (weak law of action/reaction) and sum over i :

$$\underbrace{\sum_{\substack{i,j \\ j \neq i}} \vec{F}_{ij}}_{=0} + \underbrace{\sum_i \vec{F}_i^{(e)}}_{=\vec{F}^{(e)} = \text{total external force}} = \frac{d^2}{dt^2} \sum_i m_i \vec{r}_i$$

Center of mass (CM): $\vec{R} = \dfrac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \dfrac{1}{M} \sum_i m_i \vec{r}_i$

$\Rightarrow M\ddot{\vec{R}} = \vec{F}^{(e)}$; CM moves as if all mass in CM

Total (linear) momentum: $\vec{P} = \sum_i m_i \dot{\vec{r}}_i = M\dot{\vec{R}}$

$\Rightarrow$ Conservation law : $\vec{F}^{(e)} = 0 \Rightarrow \vec{P}$ is conserved

Total angular momentum: $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$

$$\Rightarrow \dot{\vec{L}} = \sum_i \dot{\vec{r}}_i \times \vec{p}_i \quad \left(\text{since } \sum_i \dot{\vec{r}}_i \times \vec{p}_i = 0\right)$$

$$\overset{N2}{=} \underbrace{\sum_i \vec{r}_i \times \vec{F}_i^{(e)}}_{\substack{= \text{ external} \\ \text{torque } \vec{\tau}^{(e)}}} + \sum_{\substack{i,j \\ i \neq j}} \vec{r}_i \times \vec{F}_{ji}$$

Double sum is sum of pairs:

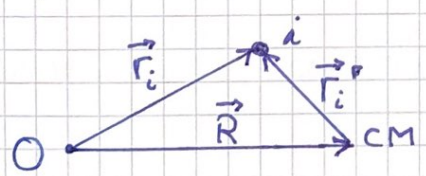
$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} \quad (\text{since } \vec{F}_{ij} = -\vec{F}_{ji})$$

If internal forces are <u>central</u>, i.e., along the joining line $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ (strong law of action/reaction), all $\vec{r}_{ij} \times \vec{F}_{ji} = 0$

$$\Rightarrow \vec{\tau}^{(e)} = \dot{\vec{L}}$$

$\Rightarrow$ Conservation law: $\vec{\tau}^{(e)} = 0 \Rightarrow \vec{L}$ is conserved

Reformulation of $\vec{L}$:



$$\vec{r}_i = \vec{R} + \vec{r}_i'$$
$$\vec{v}_i = \vec{v} + \vec{v}_i'$$

$$\Rightarrow \vec{L} = \sum_i (\vec{R} + \vec{r}_i') \times m_i (\vec{v} + \vec{v}_i')$$

Since $\sum_i m_i \vec{r}_i' = 0$, two sums are zero

$$\Rightarrow \vec{L} = \vec{R} \times M\vec{v} + \sum_i \vec{r}_i' \times \vec{p}_i'$$

orbital angular mom. relative O

internal angular mom. relative CM

Work and kinetic energy:

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i$$

$$= \sum_i \int_1^2 m_i \dot{\vec{v}}_i \cdot \vec{v}_i \, dt = \sum_i \int_1^2 d\left(\tfrac{1}{2} m_i v_i^2\right)$$

$$= \left(\sum_i \tfrac{1}{2} m_i v_i^2\right)_2 - \left(\sum_i \tfrac{1}{2} m_i v_i^2\right)_1 = T_2 - T_1$$

Reformulation of T:

$$T = \frac{1}{2} \sum_i m_i \left(\vec{v} + \vec{v}_i'\right) \cdot \left(\vec{v} + \vec{v}_i'\right)$$

$$= \frac{1}{2} \sum_i m_i v^2 + \frac{1}{2} \sum_i m_i v_i'^2 + 2 \cdot \frac{\vec{v}}{2} \cdot \frac{d}{dt} \underbrace{\sum_i m_i \vec{r}_i'}_{=0}$$

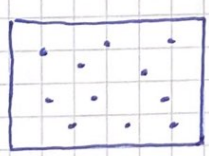
$$= \frac{1}{2} M v^2 + \frac{1}{2} \sum_i m_i v_i'^2$$

CM part

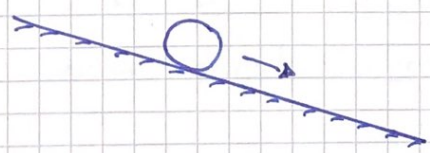
internal part
(e.g. rotation of rigid body)

1.3 Constraints and generalized coordinates

Examples of constrained motion:



gas in container



rigid body rolling on incline

Classification of constraints:

Holonomic : $f(\vec{r}_1, \vec{r}_2, \cdots, t) = 0$

Rheonomous : Time dependent

Scleronomous : Time independent

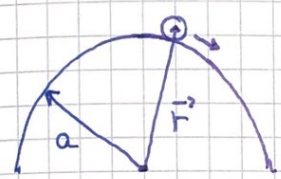
Examples :

Rolling on fixed incline : $y - ax - b = 0$

(holonomic, scleronomous)

Rolling on moving incline : holonomic, rheonomous

Rolling on spherical surface:



$r^2 - a^2 \geq 0 \quad \Rightarrow$ nonholonomic

Generalized coordinates :

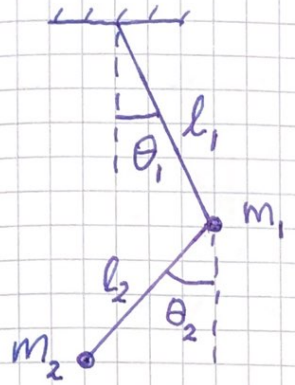
N particles has 3N independent coordinates, or degrees of freedom. With k holonomic constraints, the corresponding equations reduce this to 3N − k degrees of freedom.

New independent coordinates : $q_1, q_2, \cdots, q_{3N-k}$

Transformation equations, with constraints implicitly incorporated :

$$\vec{r}_j = \vec{r}_j(q_1, q_2, \cdots, q_{3N-k}, t) \quad ; \quad j = 1, 2, \cdots, N$$

Ex: Double pendulum (in a plane)



$N = 2$

$\Rightarrow 3N = 6$ degrees of freedom

Generalized coordinates:

θ_1, θ_2

Constraints:

$z_1 = 0 \; ; \; z_2 = 0$ (movement in xy-plane)

$$\left.\begin{aligned} x_1^2 + y_1^2 - \ell_1^2 &= 0 \\ (x_2 - x_1)^2 + (y_2 - y_1^2) - \ell_2^2 &= 0 \end{aligned}\right\} \text{(constant rod lengths)}$$

$\Rightarrow 6 - 4 = 2$ independent degrees of freedom

1.4 D'Alembert's Principle and Lagrange's Equations

We will show: For a conservative and holonomic system with n generalized coordinates q_j

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \qquad (j = 1, \dots, n)$$

$L = T - V$ = Lagrangian of the system

Proof:

Imagine infinitesimal displacements $\delta \vec{r}_i$ of the system coordinates, at a given time, and consistent with the constraints – a virtual displacement.

If equilibrium: $\vec{F}_i = 0$; $i = 1, \dots, N$

$$\Rightarrow \sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0 \quad \Rightarrow \sum_i \left(\underbrace{\vec{F}_i^{(a)}}_{\text{applied (external) force}} + \underbrace{\vec{f}_i}_{\text{force of constraint}} \right) \cdot \delta \vec{r}_i = 0$$

Assume $\sum_i \vec{f}_i \cdot \delta \vec{r}_i = 0$; <u>not</u> OK if $\vec{f}$ is sliding friction, but OK for frictionless sliding on a surface ($\vec{N} \perp \delta \vec{r}$) and rolling without sliding (contact point momentarily at rest).

$\Rightarrow$ We drop superscript (a), i.e., $\vec{F}_i$ = applied force. Constraint forces are eliminated, and the net virtual work is zero in equilibrium:

$$\sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0$$

Out of equilibrium : $\vec{F}_i - \dot{\vec{p}}_i = 0$

$$\Rightarrow \sum_i \left(\vec{F}_i - \dot{\vec{p}}_i\right) \cdot \delta\vec{r}_i = 0 \qquad \text{[D'Alemberts principle]}$$

Assume holonomic system and introduce independent coordinates q_j :

$$\vec{r}_i = \vec{r}_i\,(q_1, \dots, q_n, t) = \vec{r}_i\,(q_j, t)$$

Particle velocities :

$$\vec{v}_i = \dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\,\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} = \vec{v}_i\,(q_j, \dot{q}_j, t)$$

Virtual displacements :

$$\delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j \qquad [\text{Fixed time, } \delta t = 0]$$

1. sum :

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_{i,j} \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j = \sum_j Q_j\,\delta q_j$$

Generalized force : $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$

2. sum :

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j$$

Product rule $\Rightarrow$ $m_i \ddot{\vec{r}}_i \cdot \partial\vec{r}_i / \partial q_j = \frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i \vec{v}_i \cdot \frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_j}$

Can $\frac{d}{dt}$ and $\frac{\partial}{\partial q_j}$ be interchanged in the last term? Yes :

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_j} &= \sum_k \left(\frac{\partial}{\partial q_k}\frac{\partial \vec{r}_i}{\partial q_j}\right)\dot{q}_k + \frac{\partial}{\partial t}\frac{\partial \vec{r}_i}{\partial q_j} \\ \frac{\partial}{\partial q_j}\underbrace{\frac{d\vec{r}_i}{dt}}_{\vec{v}_i} &= \frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k}\,\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right) \end{aligned}\right\} \text{OK!}$$

We also need $\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$ [Follows from $\vec{v}_i = \ldots$ on p.9]

This yields:

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left\{ \frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right\}$$

$$= \sum_i \left\{ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left(\tfrac{1}{2} m_i v_i^2 \right) - \frac{\partial}{\partial q_j} \left(\tfrac{1}{2} m_i v_i^2 \right) \right\}$$

$$= \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \qquad \left[\text{since } T = \sum_i \tfrac{1}{2} m_i v_i^2 \right]$$

D'Alembert's principle becomes:

$$\sum_j \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

With holonomic constraints and independent δq_j, each term in this sum must vanish:

$$\boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \quad ; \quad j = 1, 2, \ldots, n$$

If all forces are conservative, i.e., $\vec{F}_i = -\nabla_i V$:

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\left[\frac{\partial V}{\partial q_j} = \sum_i \left(\frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i} \frac{\partial z_i}{\partial q_j} \right) = \sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right]$$

We assume (for now) that V is independent of $\dot{q}_j$

$$\Rightarrow \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} (T - V)$$

$$\Rightarrow \boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad ; \quad L = T - V} \qquad \text{(qed)}$$

Scalar equations, <u>easier</u> than equations with vectors $\vec{F}_i$ and $\dot{\vec{p}}_i$

1.6 Simple examples of the Lagrangian formulation

Ex. 1: Single particle

a) Cartesian coordinates

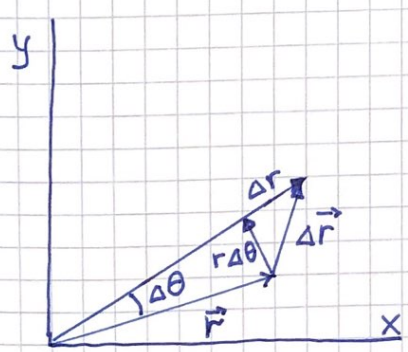
$$q_1 = x, \quad q_2 = y, \quad q_3 = z$$
$$Q_1 = F_x, \quad Q_2 = F_y, \quad Q_3 = F_z$$
$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$\frac{\partial T}{\partial q_j} = 0, \quad \frac{\partial T}{\partial \dot{q}_j} = m \dot{q}_j \quad ; \quad j = 1, 2, 3$$
$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \quad ; \quad -''-$$

$$\Rightarrow m\ddot{x} = F_x, \quad m\ddot{y} = F_y, \quad m\ddot{z} = F_z \quad \text{(N2, as expected)}$$

b) Plane polar coordinates

$$\left.\begin{aligned} x &= r \cos\theta \\ y &= r \sin\theta \end{aligned}\right\} \Rightarrow \{q_j\} = \{r, \theta\}$$

$$\dot{x} = \dot{r} \cos\theta - r\dot{\theta} \sin\theta$$
$$\dot{y} = \dot{r} \sin\theta + r\dot{\theta} \cos\theta$$
$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$



$$\Delta \vec{r} = \Delta r \, \hat{r} + r \Delta\theta \, \hat{\theta}$$

Radial velocity: $\frac{\Delta r}{\Delta t} \to \dot{r}$

Azimuth $-''-$: $\frac{r\Delta\theta}{\Delta t} \to r\dot{\theta}$

Generalized force components: $Q_j = \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_j}$

$$Q_r = \vec{F} \cdot \frac{\partial \vec{r}}{\partial r} = \vec{F} \cdot \hat{r} = F_r$$

$$Q_\theta = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \theta} = \vec{F} \cdot r\hat{\theta} = r F_\theta$$

Lagrange's equations:

$$q_1 = r, \quad \frac{\partial T}{\partial r} = m r \dot{\theta}^2, \quad \frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{r}} = m\ddot{r}$$

$$\Rightarrow \quad m\ddot{r} - \underbrace{m r \dot{\theta}^2}_{\text{centripetal acceleration}} = F_r$$

$$q_2 = \theta, \quad \frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = m r^2 \dot{\theta}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta}$$

$$\Rightarrow \quad m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta} = r F_\theta$$

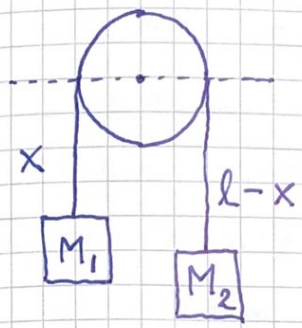
This is angular momentum, torque and N2 for rotation:

$$\vec{L} = \vec{r} \times \vec{p} \Rightarrow L = r p_\theta = r m v_\theta = r m r \dot{\theta} = m r^2 \dot{\theta}$$

$$\vec{\tau} = \vec{r} \times \vec{F} \Rightarrow \tau = r F_\theta$$

$$\Rightarrow \frac{d}{dt}(m r^2 \dot{\theta}) = r F_\theta \quad \text{or} \quad \frac{dL}{dt} = \tau$$

Ex 2: Atwood's machine



$\Rightarrow$ only 1 independent coord: x

$$V = -M_1 g x - M_2 g (l - x)$$

$$T = \frac{1}{2}(M_1 + M_2)\dot{x}^2$$

$$L = T - V$$

$$\frac{\partial L}{\partial x} = (M_1 - M_2) g \,, \quad \frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\dot{x}$$

$$\Rightarrow \quad (M_1 + M_2)\ddot{x} - (M_1 - M_2) g = 0$$

$$\Rightarrow \quad \ddot{x} = \frac{M_1 - M_2}{M_1 + M_2} g$$

The constraint force, here the tension in the rope, does not appear in the Lagrangian formulation, as expected.

1.5 Generalized potentials

So far, we have assumed potentials $V(q_j)$ dependent on <u>coordinates</u> only. We would like to use the Lagrangian formulation also when the forces depend on the particle <u>velocity</u>, e.g., electromagnetism and fluid friction.

Electromagnetism:

First, note that Lagrange's equations have the same form if the generalized forces can be expressed by a generalized potential $U(q_j, \dot{q}_j)$ as

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j}$$

Then, with $L = T - U$ and $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$,

we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \qquad \text{as before.}$$

Maxwells equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ; \quad \nabla \cdot \vec{D} = \rho$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \quad ; \quad \nabla \cdot \vec{B} = 0$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \quad ; \quad \vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

In absence of magnetic or dielectric materials:

$$\vec{M} = 0, \quad \vec{P} = 0$$

$$\Rightarrow \nabla \cdot \vec{E} = \rho/\varepsilon_0 \quad ; \quad \nabla \times \vec{B} = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Lorentz force on moving charge: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Scalar potential ϕ and vector potential $\vec{A}$:

$\nabla \cdot \vec{B} = 0 \Rightarrow$ may write $\vec{B} = \nabla \times \vec{A}$

(since $\nabla \cdot (\nabla \times \vec{V}) = 0$ for any vector $\vec{V}$)

$$\Rightarrow \nabla \times \vec{E} + \frac{\partial}{\partial t}(\nabla \times \vec{A}) = \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$\Rightarrow$ may write $\vec{E} + \partial \vec{A}/\partial t = -\nabla \phi$

(since $\nabla \times \nabla f = 0$ for any scalar f)

Lorentz force in terms of these potentials :

$$\vec{F} = q\left(-\nabla \phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \nabla \times \vec{A}\right)$$

Intermezzo on notation :

Einsteins summation convention :
A repeated index should implicitly be summed over all its possible values.

Assume cartesian coordinates with

$$x_1 = x, \quad x_2 = y, \quad x_3 = z$$

Then,

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$
$$= \sum_{i=1}^{3} A_i B_i$$
$$= A_i B_i$$

with summation over i assumed, without the Σ sign.

Other examples:

$$\nabla \cdot \vec{A} = \frac{\partial A_i}{\partial x_i} = \partial_i A_i$$

$$\nabla \phi = \hat{x}_i \, \partial_i \phi$$

Levi-Civita tensor:

Useful when dealing with cross products.

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } i,j,k \text{ cyclic} \quad (\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1) \\ -1 & \text{— " — anticyclic} \quad (\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1) \\ 0 & \text{if 2 or 3 indices are equal} \end{cases}$$

Then:

$$\vec{A} = \vec{B} \times \vec{C} \; : \quad A_i = \varepsilon_{ijk} B_j C_k$$

$$\text{Ex}: \; A_1 = \varepsilon_{123} B_2 C_3 + \varepsilon_{132} B_3 C_2$$
$$= B_2 C_3 - B_3 C_2 \qquad \text{(OK)}$$

$$\vec{B} = \nabla \times \vec{A} \; : \quad B_i = \varepsilon_{ijk} \partial_j A_k$$

Often useful: $\varepsilon_{ijk} \varepsilon_{i\ell m} = \delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{k\ell}$

$$\text{Ex}: \; (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \times \vec{B})_i \, (\vec{C} \times \vec{D})_i$$

$$= \varepsilon_{ijk} A_j B_k \, \varepsilon_{i\ell m} C_\ell D_m$$

$$= (\delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{k\ell}) A_j B_k C_\ell D_m$$

$$= A_j C_j B_k D_k - A_j D_j B_k C_k$$

$$= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

[More in assignments.]

We had: $\vec{F} = q\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times\nabla\times\vec{A}\right)$

$$\Rightarrow F_i = q\left\{-\partial_i\phi - \partial_t A_i + \left[\vec{v}\times(\nabla\times\vec{A})\right]_i\right\}$$

$$\left[\vec{v}\times(\nabla\times\vec{A})\right]_i = \varepsilon_{ijk}\, v_j\, \varepsilon_{klm}\, \partial_l A_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, v_j\, \partial_l A_m = v_j\,\partial_i A_j - v_j\,\partial_j A_i$$

1. term: $v_j\,\partial_i A_j = \partial_i\, v_j A_j = \partial_i\, \vec{v}\cdot\vec{A}$ (since $\partial_i v_j = 0$)

2. term: $v_j\,\partial_j A_i = \sum_j \frac{\partial A_i}{\partial x_j}\underbrace{\frac{dx_j}{dt}}_{v_j} = \frac{dA_i}{dt} - \frac{\partial A_i}{\partial t}$

$$\Rightarrow F_i = q\left\{-\partial_i\phi - \partial_t A_i + \partial_i\,\vec{v}\cdot\vec{A} - \left(\frac{dA_i}{dt} - \partial_t A_i\right)\right\}$$

$$= q\left\{-\partial_i\left[\phi - \vec{v}\cdot\vec{A}\right] + \frac{d}{dt}\frac{\partial}{\partial v_i}\left[\phi - \vec{v}\cdot\vec{A}\right]\right\}$$

(since $\partial\phi/\partial v_i = 0$ and $\partial(\vec{v}\cdot\vec{A})/\partial v_i = A_i$)

Now, F_i is on the desired form,

$$F_i = -\frac{\partial U}{\partial x_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}_i}$$

with $U(x_i, \dot{x}_i) = q\phi - q\,\vec{A}\cdot\vec{v}$

Lagrangian for charged particle in E.M. field:

$$L = T - U = \frac{1}{2}mv^2 - q\phi + q\,\vec{A}\cdot\vec{v}$$

Friction:

May write Lagrange's equations on the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

with

$L = T - V$; V = potential due to conservative forces

Q_i = forces not expressed by a potential

Assume fluid friction and laminar flow, i.e., a frictional force prop. with the particle velocity:

$$F_x = -k_x v_x$$

May be expressed as $F_x = -\frac{\partial}{\partial v_x}\frac{1}{2}k_x v_x^2$, or in 3D $\vec{F} = -\nabla_v \mathcal{F}$ with

$$\mathcal{F} = \frac{1}{2}\sum_i \left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right)$$

= Rayleigh's dissipation function

Work done by the system (all particles i) against friction:

$$dW = -\sum_i \vec{F}_i \cdot d\vec{r}_i = -\sum_i \vec{F}_i \cdot \vec{v}_i\, dt = \sum_i \left(k_x^2 v_{ix}^2 + \dots\right) dt$$

$\Rightarrow dW/dt = 2\mathcal{F}$ = dissipated power due to friction

Generalized frictional force:

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \nabla_v \mathcal{F} \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

Ex: Stokes' law $\vec{F} = -6\pi\eta a \vec{v}$ for sphere of radius a moving in fluid with viscosity η

2. Variational principles and Lagrange's equations

2.1 Hamilton's principle

We derived Lagrange's equations from D'Alembert's differential principle, by considering small virtual displacements from a given state of the system.

Next, we will derive Lagrange's equations from an integral (global) principle, by considering small variations in the total movement of the system.

The state of the system at a time t is given by a point in a configuration space with axes corresponding to the generalized coordinates $q_1 \dots q_n$ $(n = 3N - k)$. The movement of the system is then described by a curve in this space.

Hamilton's principle : The system moves from t_1 to t_2 such that the action

$$I = \int_{t_1}^{t_2} L \, dt \qquad (L = T - V = \text{Lagrangian})$$

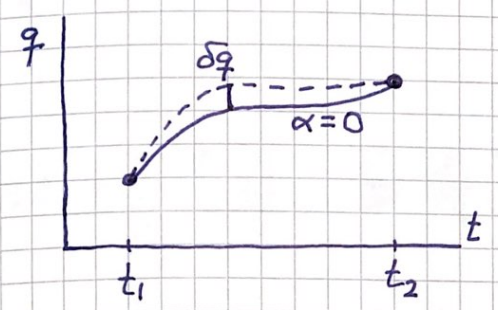
has a stationary value for the actual path of the motion, i.e.,

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1 \cdots q_n ; \dot{q}_1 \cdots \dot{q}_n ; t) \, dt = 0$$

(for fixed t_1 and t_2)

2.3 Lagrange's equations from Hamilton's principle

Assume one degree of freedom $q = q(t)$



Paths near actual path are parametrized with parameter α.
Actual path: $\alpha = 0$, $\delta I = 0$

$$\Rightarrow \quad q(t,\alpha) = q(t,0) + \underbrace{\alpha\,\eta(t)}_{\delta q(t)} \qquad (\eta(t_1) = \eta(t_2) = 0)$$

(otherwise arbitrary $\eta(t)$)

$$I(\alpha) = \int_{t_1}^{t_2} L\left[q(t,\alpha),\ \dot{q}(t,\alpha),\ t\right] dt$$

Virtual variations at fixed time t:

$$\delta q = \left(\frac{\partial q}{\partial \alpha}\right)_0 d\alpha$$

$$\delta \dot{q} = \left(\frac{\partial \dot{q}}{\partial \alpha}\right)_0 d\alpha$$

Variation of the action:

$$\delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right] dt$$

In $\delta \dot{q} = \delta \frac{d}{dt} q$, δ and $\frac{d}{dt}$ may be interchanged:

$$\delta \dot{q} = \left(\frac{\partial \dot{q}}{\partial \alpha}\right)_0 d\alpha = \frac{\partial^2 q}{\partial \alpha \partial t}\, d\alpha = \dot{\eta}\, d\alpha$$

$$\frac{d}{dt}\delta q = \frac{d}{dt}\left(\frac{\partial q}{\partial \alpha}\right)_0 d\alpha = \frac{d}{dt}\eta\, d\alpha = \dot{\eta}\, d\alpha$$

$$\Rightarrow \quad \delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q\right] dt$$

Integration by parts on 2. term:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q \, dt = \underbrace{\Big|_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \delta q}_{=0} - \int_{t_1}^{t_2} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}}\right) \delta q \, dt$$

since $\delta q = 0$ in t_1 and t_2

$$\Rightarrow \delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right] \delta q \, dt = 0$$

Since δq is arbitrary: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$

Generalization to n degrees of freedom:

$$\delta I = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \, dt = 0$$

$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad ; \quad i = 1, \dots, n$$

Valid when q_i are independent, i.e., with holonomic constraints.

Valid for conservative systems, $V = V(q_i)$, but also for non-conservative systems when the gen. forces can be expressed as

$$Q_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt} \frac{\partial U}{\partial \dot{q}_i} \qquad (L = T - U)$$

I.e., valid for socalled monogenic systems.

2.2 Calculus of variations

Variational principle not limited to the Lagrangian and corresponding action integral.

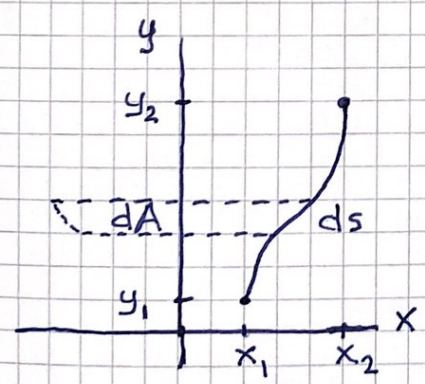
Consider a curve $y = y(x)$ between $y_1 = y(x_1)$ and $y_2 = y(x_2)$; $y' = dy/dx$. We want to find a stationary value (minimum or maximum) of the integral

$$I = \int_{x_1}^{x_2} f(y, y', x)\, dx$$

where f is a function defined on the curve $y(x)$. We adopt the derivation on p. 20-21, with $t \to x$, $q \to y$, $\dot{q} \to y'$ and $L(q, \dot{q}, t) \to f(y, y', x)$, and find:

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \qquad \text{(Euler's equations)}$$

Ex 1: Minimum surface of revolution



Which curve $y(x)$ between (x_1, y_1) and (x_2, y_2) gives minimum surface of revolution?

$$dA = 2\pi x \cdot ds = 2\pi x \cdot \sqrt{dx^2 + dy^2} = 2\pi x \sqrt{1 + y'^2}\, dx$$

$$\Rightarrow A = 2\pi \int_{x_1}^{x_2} x\sqrt{1 + y'^2}\, dx \qquad \Rightarrow f(y, y', x) = x\sqrt{1 + y'^2}$$

$$\frac{\partial f}{\partial y} = 0 \,, \quad \frac{\partial f}{\partial y'} = \frac{x y'}{\sqrt{1+y'^2}} \quad \Rightarrow \frac{d}{dx} \frac{x y'}{\sqrt{1+y'^2}} = 0$$

$$\Rightarrow \frac{x y'}{\sqrt{1+y'^2}} = a \quad (= \text{const.}) \quad \Rightarrow \ldots \Rightarrow \frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}}$$

$$\Rightarrow \quad y = a \int \frac{dx}{\sqrt{x^2-a^2}}$$

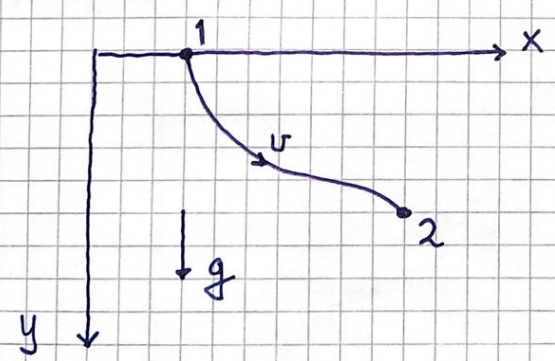
Subst. $x = a \cosh z \quad \Rightarrow \quad dx = a \sinh z \, dz$

$$x^2 - a^2 = a^2 \sinh^2 z$$

$$\Rightarrow y = a \int dz = az + b = a \operatorname{arcosh}(x/a) + b$$

or $x = a \cosh\left(\frac{y-b}{a}\right)$, where a and b are determined by $y(x_1) = y_1$ and $y(x_2) = y_2$

Ex 2: The brachistochrone problem



Which curve from 1 to 2 gives shortest time? Assume $v(1) = 0$ and energy conservation.

$$dt = ds/v \quad ; \quad ds = dx \cdot \sqrt{1+y'^2}$$

$$\frac{1}{2} m v^2 = mgy \quad \Rightarrow \quad v = \sqrt{2gy}$$

$$\Rightarrow t_{12} = \int_1^2 dt = \frac{1}{\sqrt{2g}} \int_1^2 f(y, y', x) dx \quad ; \quad f = \sqrt{\frac{1+y'^2}{y}}$$

$$\frac{\partial f}{\partial y} = -\frac{\sqrt{1+y'^2}}{2y^{3/2}} \quad ; \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{y}\sqrt{1+y'^2}}$$

$$\Rightarrow \quad \frac{d}{dx}\frac{y'}{\sqrt{y}\sqrt{1+y'^2}} + \frac{\sqrt{1+y'^2}}{2y^{3/2}} = 0$$

Carry out d/dx using product and chain rule.
Cancel common factor $y^{-1/2}(1+y'^2)^{-1/2}$.
Multiply with y'. Use rules for logarithm.

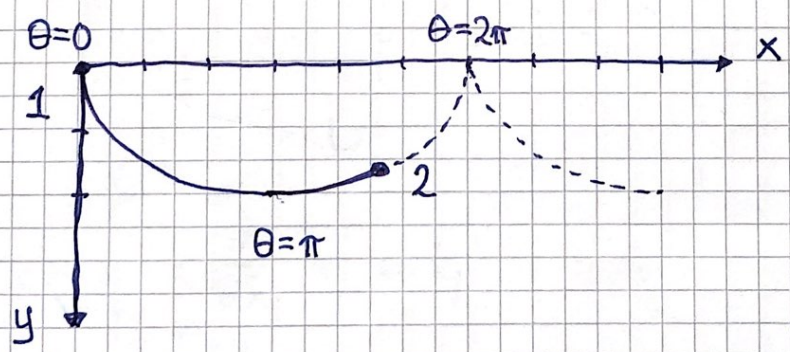
$$\Rightarrow \frac{d}{dx}\left\{\ln\left[y(1+y'^2)\right]\right\} = 0 \quad \Rightarrow \quad y(1+y'^2) = 2k \quad (= \text{const.})$$

Parametric solution: $x = k(\theta - \sin\theta)$; $y = k(1-\cos\theta)$

[Check: $\frac{dy}{dx} = \frac{k\sin\theta\, d\theta}{k(1-\cos\theta)d\theta} = \frac{\sin\theta}{1-\cos\theta}$

$$\Rightarrow 1+y'^2 = \ldots = \frac{2}{1-\cos\theta} \quad \Rightarrow \quad y(1+y'^2) = 2k \ ; \ \text{OK}]$$

Assume $x_1 = y_1 = 0$:



Cycloid (same as path followed by point on rolling wheel)

2.6 Conservation laws and symmetry properties

Consider first a system of point masses m_i in a potential V only dependent on positions. Then:

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \sum_j \frac{1}{2} m_j (\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2) = m_i \dot{x}_i = p_{ix}$$

With <u>generalized</u> coordinates q_i, one defines the generalized (or canonical, or conjugate) momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

If the potential depends on velocities, $U = U(q_i, \dot{q}_i)$, canonical momentum and mechanical momentum are different.

Ex: Particle in E.M. field

$$U = q\phi - q\vec{A}\cdot\vec{v} = q\phi - qA_i\dot{x}_i$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}_i\dot{x}_i$$

$$L = T - U$$

$$\Rightarrow \underbrace{p_j}_{\text{canonical mom.}} = \frac{\partial L}{\partial \dot{x}_j} = \underbrace{m\dot{x}_j}_{\text{mechanical mom.}} + qA_j$$

<u>Cyclic</u> coord. q_i if L is independent of q_i

Then: $\frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt} p_i = 0 \Rightarrow p_i = \text{const.}$

$\Rightarrow$ The canonical momentum corresponding to a cyclic coordinate is conserved.

Ex: Particle in EM field with $\phi, \vec{A}$ indep. of x

$\Rightarrow L$ indep. of $x \quad \Rightarrow \quad x$ is cyclic coord.

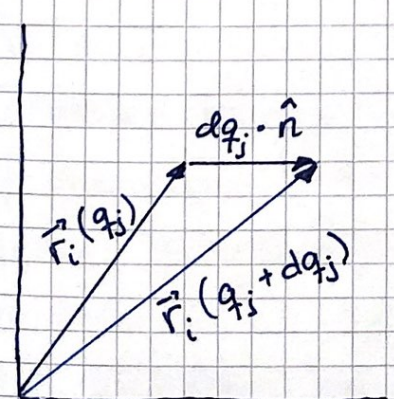
$\Rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x = \text{const.}$

(but mech. mom. $m\dot{x}$ is not conserved)

Let us take a closer look at conservation laws related to translation and rotation, as well as conservation of energy.

Translation:

Consider gen. coord q_j such that dq_j implies translation of the whole system in a direction $\hat{n}$



$$\frac{\partial \vec{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\vec{r}_i(q_j + dq_j) - \vec{r}_i(q_j)}{dq_j}$$

$$= \frac{dq_j \cdot \hat{n}}{dq_j} = \hat{n}$$

(for all particles i)

Assume conservative system, $V = V(q)$.
Clearly, $\partial T/\partial q_j = 0$. (cf. Ex 1a on p. 11; velocities, and hence T, unaffected by shifting the origin)

Lagrange eqn: $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = Q_j$ or: $\dot{p}_j = Q_j$

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

$$\underset{(p.10)}{=} \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \left(\sum_i m_i \vec{v}_i\right)\cdot \hat{n} = \vec{P}\cdot\hat{n}$$

= component of total lin. mom. $\vec{P}$ along $\hat{n}$

$$Q_j \overset{(p.10)}{=} \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \left(\sum_i \vec{F}_i\right)\cdot\hat{n} = \vec{F}\cdot\hat{n}$$

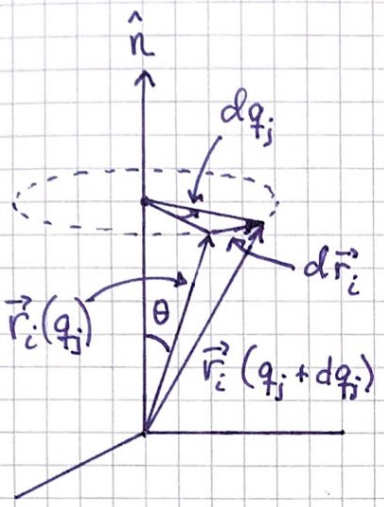
= comp. of total force $\vec{F}$ along $\hat{n}$

$\Rightarrow$ Here, $\dot{p}_j = Q_j$ is eqn. of motion for total linear momentum.

If q_j is cyclic, i.e., $\frac{\partial L}{\partial q_j} = -\frac{\partial V}{\partial q_j} = Q_j = 0$

$\Rightarrow \dot{p}_j = 0$; conservation law for linear momentum

Rotation:
Consider gen. coord. q_j such that dq_j implies rotation of the whole system around an axis $\hat{n}$.
Again, $\partial T/\partial q_j = 0$ (cf. Ex 1b on p. 11-12), and we will again look closer at $\dot{p}_j = Q_j$ with
$p_j = \partial T/\partial \dot{q}_j$ and $Q_j = \sum_i \vec{F}_i \cdot \partial \vec{r}_i/\partial q_j$



$$|d\vec{r}_i| = r_i \sin\theta \, dq_j$$

$$\Rightarrow \left|\frac{\partial \vec{r}_i}{\partial q_j}\right| = r_i \sin\theta$$

$$d\vec{r}_i \perp \vec{r}_i \quad \text{and} \quad d\vec{r}_i \perp \hat{n}$$

$$\Rightarrow \frac{\partial \vec{r}_i}{\partial q_j} = \hat{n} \times \vec{r}_i$$

We will need: $\vec{a}\cdot(\vec{b}\times\vec{c}) = a_i \varepsilon_{ijk} b_j c_k$

$$= b_j \varepsilon_{jki} c_k a_i$$

$$= \vec{b}\cdot(\vec{c}\times\vec{a})$$

$$\Rightarrow p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i m_i \vec{v}_i \cdot (\hat{n}\times\vec{r}_i)$$

$$= \sum_i \hat{n}\cdot(\vec{r}_i \times m_i\vec{v}_i) = \hat{n}\cdot\sum_i \vec{L}_i = \hat{n}\cdot\vec{L}$$

= comp. of angular mom. $\vec{L}$ along $\hat{n}$

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \vec{F}_i \cdot (\hat{n}\times\vec{r}_i) = \hat{n}\cdot\sum_i \vec{r}_i\times\vec{F}_i$$

$$= \hat{n}\cdot\sum_i \vec{\tau}_i = \hat{n}\cdot\vec{\tau}$$

= comp. of total torque $\vec{\tau}$ along $\hat{n}$

So, now $\dot{p}_j = Q_j$ is eqn. of motion for total angular momentum.

If q_j is cyclic, $\partial L/\partial q_j = -\frac{\partial V}{\partial q_j} = Q_j = 0$

$\Rightarrow \dot{p}_j = 0$; <u>conservation law for angular momentum</u>

In summary: Translational invariance of the system yields conservation of linear momentum; rotational invariance yields conservation of angular momentum.

2.7 Conservation of energy

Assume $L = L(q_i, \dot{q}_i, t)$ with $V = V(q_i)$. Consider $\frac{dL}{dt}$:

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i} \frac{dq_i}{dt} + \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} + \frac{\partial L}{\partial t}$$

In 1. sum, Lagrange's eqns. gives $\frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}$

$\Rightarrow$ With product rule, the two sums are $\sum_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)$

$$\Rightarrow \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right) + \frac{\partial L}{\partial t} = 0$$

The energy function: $H(q, \dot{q}, t) = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$

Then, if $\partial L / \partial t = 0$: $dH/dt = 0$

Let us show that (usually!) H equals the total energy, $T+V$.
Assume that T contains only quadratic terms in the generalized velocities, i.e., terms prop. to $\dot{q}_i^2$ or $\dot{q}_i \dot{q}_j$; this is usually the case. Then, T is said to be a homogeneous function of the 2. degree, and

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

$$\Rightarrow \quad H = 2T - L = 2T - (T - V) = T + V$$

$\Rightarrow$ If $\frac{\partial L}{\partial t} = 0$, the total energy is conserved

8. Hamilton's equations

"Lagrange vs Hamilton":

- Equivalent equations and the same physics; just different methods.
- Hamilton's procedure suitable within quantum and statistical mechanics.

We assume holonomic systems, and potentials that are conservative, $V = V(q)$, or velocity dependent, $U = U(q, \dot{q})$, such that $Q_i = -\partial U/\partial q_i + \frac{d}{dt}\partial U/\partial \dot{q}_i$, to include E.M. fields in our description.

- Lagrange: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$; $i = 1, 2, \cdots, n$
  I.e., n 2. order differential equations.
  State of system: Point in n-dim. configuration space with axes q_i

- Hamilton: $2n$ 1. order diff. eqns.
  State of system: Point in $2n$-dim. phase space with axes q_i and p_i, with

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad ; \quad i = 1, 2, \cdots, n$$

q, p are <u>canonical variables</u>

- Both formulations result in $2n$ integration constants and require $2n$ initial conditions for complete solution.

8.1 Legendre transformations and Hamiltons eqns of motion

Transition from Lagrange to Hamilton involves a <u>change of variables</u>, from $(q, \dot{q}, t)$ to (q, p, t) with $p_i = \partial L / \partial \dot{q}_i$.

Legendre transformation:

Assume function $f(x,y)$ such that

$$df = u \cdot dx + v \cdot dy \quad ; \quad u(x,y) = \frac{\partial f}{\partial x} \; , \quad v(x,y) = \frac{\partial f}{\partial y}$$

Want to change variables ("basis") from (x,y) to (u,y).
Define $g = f - u \cdot x$ (old function minus product of new and old variable). Then:

$$dg = df - u dx - x du = v dy - x du$$

with $v = v(u,y) = \frac{\partial g}{\partial y}$; $x = x(u,y) = -\frac{\partial g}{\partial u}$

Ex: Thermodynamics

1. + 2. law : $TdS = dU + pdV \Rightarrow dU = TdS - pdV$

$U = U(S,V)$, $T = \partial U / \partial S$, $p = -\partial U / \partial V$

More convenient with enthalpy $H = H(S,p) = U + pV$ for isentropic and isobaric processes:

$dH = dU + pdV + Vdp = TdS + Vdp$; $T = \partial H / \partial S$, $V = \partial H / \partial p$

More convenient with Gibbs' free energy $G(T,p) = H - TS$ for isothermal and isobaric processes:

$dG = dH - TdS - SdT = Vdp - SdT$; $V = \partial G / \partial p$, $S = -\partial G / \partial T$

The natural Legendre transf. for changing variables from $(q, \dot{q}, t)$ to (q, p, t) is:

$$H = p\dot{q} - L = H(q, p, t)$$

With summation over repeated index:

$$dH = d(p_i \dot{q}_i - L) = p_i \, d\dot{q}_i + \dot{q}_i \, dp_i - dL$$

In addition, with $H = H(q_i, p_i, t)$:

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

We have also:

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

Canonical momentum: $p_i = \partial L / \partial \dot{q}_i$

Lagrange's equations:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt} p_i = \dot{p}_i$$

The terms with $d\dot{q}_i$ cancel

$$\Rightarrow dH = -\dot{p}_i \, dq_i + \dot{q}_i \, dp_i - \frac{\partial L}{\partial t} dt$$

Comparison $(dH = dH)$ yields:

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

Hamilton's (canonical) equations of motion

and $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

[whereas $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$ for the energy function $H(q, \dot{q}, t)$]

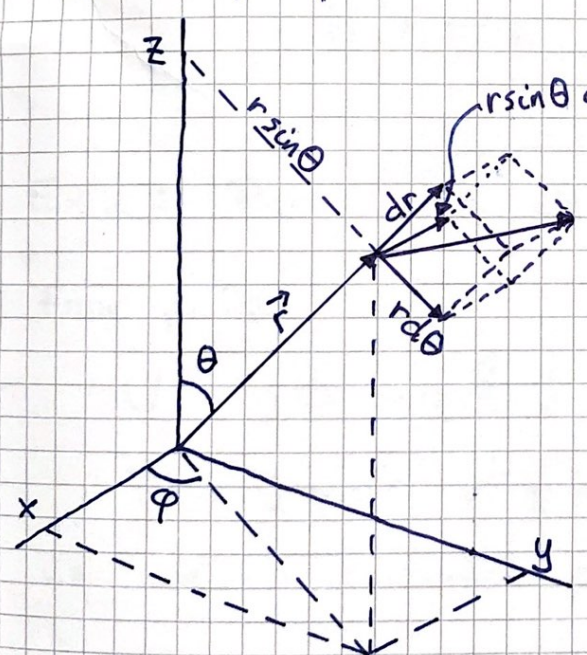
In summary, the typical recipe in the Hamilton(ian) formalism:

1. Construct $L(q, \dot{q}, t)$
2. Define canonical momenta $p_i = \partial L / \partial \dot{q}_i$
3. Construct $H = p_i \dot{q}_i - L = H(q, \dot{q}, p, t)$
4. Use $p_i = \partial L / \partial \dot{q}_i$ to express $\dot{q} = \dot{q}(q, p, t)$
5. Eliminate $\dot{q}_i$ from H to arrive at $H(q, p, t)$
6. Use H to solve the canonical equations of motion

Two important examples:

Ex 1: Particle in central forcefield, $V = V(r)$

Reminder, spherical coordinates: $q_i = (r, \theta, \varphi)$



$$d\vec{r} = \hat{r}\, dr + \hat{\theta}\, r\, d\theta + \hat{\varphi}\, r \sin\theta\, d\varphi$$

$$\Downarrow$$

$$\dot{\vec{r}} = \hat{r}\, \dot{r} + \hat{\theta}\, r \dot{\theta} + \hat{\varphi}\, r \sin\theta \cdot \dot{\varphi}$$

$$\Downarrow$$

$$T = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \cdot \dot{\varphi}^2 \right)$$

Canonical momenta:

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}, \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \sin^2\theta \cdot \dot{\varphi}$$

Write down Hamiltonian:

$$H = p_i \dot{q}_i - L = p_i \dot{q}_i - T + V$$

$$= m\dot{r}\dot{r} + mr^2\dot{\theta}\dot{\theta} + mr^2\sin^2\theta\,\dot{\varphi}\dot{\varphi}$$

$$- \tfrac{1}{2}m\dot{r}^2 - \tfrac{1}{2}mr^2\dot{\theta}^2 - \tfrac{1}{2}mr^2\sin^2\theta\,\dot{\varphi}^2 + V(r)$$

$$= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) + V(r)$$

$$= T + V$$

Eliminate $\dot{q}_i$:

$$\dot{r} = p_r/m, \quad \dot{\theta} = p_\theta/mr^2, \quad \dot{\varphi} = p_\varphi/mr^2\sin^2\theta$$

$$\Rightarrow H(q,p) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) + V(r)$$

$$= H(r, \theta, p_r, p_\theta, p_\varphi)$$

Ex 2: Particle in electromagnetic field

From p. 25:

$$U = q\Phi - q\vec{A}\cdot\vec{v} = q\Phi - qA_i\dot{x}_i \quad \text{(sum over } i\text{)}$$

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\dot{x}_i\dot{x}_i$$

i.e., we use cartesian coordinates, $x_1 = x$ etc.

$$L = T - U = \tfrac{1}{2}m\dot{x}_i\dot{x}_i + qA_i\dot{x}_i - q\Phi$$

$$p_i = \partial L/\partial x_i = m\dot{x}_i + qA_i \quad \Rightarrow \dot{x}_i = \frac{1}{m}(p_i - qA_i)$$

$$H = p_i \dot{x}_i - L$$

$$= (m\dot{x}_i + qA_i)\dot{x}_i - \frac{1}{2} m \dot{x}_i \dot{x}_i - qA_i \dot{x}_i + q\phi$$

$$= \frac{1}{2} m \dot{x}_i \dot{x}_i + q\phi$$

= mechanical energy + potential energy

= total energy

But: $H \neq T + U$ due to velocity dependence of U

We eliminate $\dot{x}_i$:

$$H(x_i, p_i, t) = \frac{1}{2m}(p_i - qA_i)(p_i - qA_i) + q\phi$$

$$= \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi$$

$\vec{p}$ = canonical momentum

$\vec{p} - q\vec{A}$ = mechanical momentum

$\phi = \phi(x_i, t)$, $\vec{A} = \vec{A}(x_i, t)$

If $\vec{A}$ and ϕ are time independent:

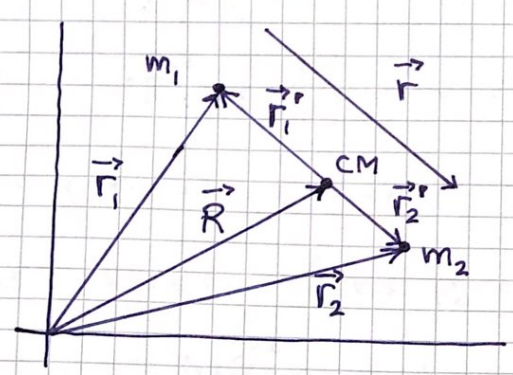
$$\frac{\partial L}{\partial t} = 0$$

and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0$$

3. The Central Force Problem

3.1 Reduction to a one-body problem



$N = 2$

$\Rightarrow 3N = 6$ degrees of freedom

We choose as generalized coordinates (vectors)

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad \text{and} \quad \vec{r} = \vec{r}_2 - \vec{r}_1$$

Assume conservative system and central forces:

$$V = V(r) \; ; \quad r = |\vec{r}|$$

Lagrangian: $L = T - V$

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} = \frac{1}{2} m_1 (\dot{\vec{R}} + \dot{\vec{r}}_1')^2 + \frac{1}{2} m_2 (\dot{\vec{R}} + \dot{\vec{r}}_2')^2$$

$$= \frac{1}{2}(m_1 + m_2) \dot{\vec{R}}^2 + \underbrace{\frac{1}{2} m_1 \dot{\vec{r}}_1'^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2'^{\,2}}_{T'} + \dot{\vec{R}} \cdot (m_1 \dot{\vec{r}}_1' + m_2 \dot{\vec{r}}_2')$$

Clearly, $\underline{m_1 \vec{r}_1' + m_2 \vec{r}_2' = 0}$ ("CM relative to CM")

$$\Rightarrow \frac{d}{dt}(m_1 \vec{r}_1' + m_2 \vec{r}_2') = 0$$

$$\Rightarrow T = \frac{1}{2} M \dot{\vec{R}}^2 + T' \; ; \; M = m_1 + m_2$$

Also, $\underline{\vec{r} = \vec{r}_2' - \vec{r}_1'}$ $\Rightarrow \vec{r}_1' = -\frac{m_2 \vec{r}}{M}$, $\vec{r}_2' = \frac{m_1 \vec{r}}{M}$

$$\Rightarrow T^r = \frac{m_1 m_2}{2M} \dot{\vec{r}}^{\,2}$$

The reduced mass: $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \Rightarrow \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$

$$\Rightarrow T^r = \frac{1}{2}\mu \dot{\vec{r}}^{\,2}$$

$$\Rightarrow L = \frac{1}{2}M\dot{\vec{R}}^{\,2} + \frac{1}{2}\mu\dot{\vec{r}}^{\,2} - V(r) = L(r, \dot{\vec{r}}, \dot{\vec{R}})$$

$\Rightarrow \vec{R}$ is a cyclic coordinate

$\Rightarrow \dot{\vec{R}}$ is constant (or: $p_R = \frac{\partial L}{\partial \dot{R}} = M\dot{R} = \text{const.}$)

$\Rightarrow$ CM term $\frac{1}{2}M\dot{\vec{R}}^{\,2}$ can be ignored

$\Rightarrow$ Two-body problem is reduced to one-body problem, with Lagrangian

$$L = T - V = \frac{1}{2}\mu\dot{\vec{r}}^{\,2} - V(r)$$

3.2 Equations of motion

So, we consider a mass m in a central forcefield, with $V = V(r)$. Rotational invariance (around any axis) $\Rightarrow$ Angular momentum is conserved:

$$\vec{L} = \vec{r} \times \vec{p} = \text{constant}$$

Only possible if $\vec{r}$ and $\vec{p}$ lie in a plane $\perp \vec{L}$

$\Rightarrow$ We use polar coordinates (r, θ).

$$\Rightarrow L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

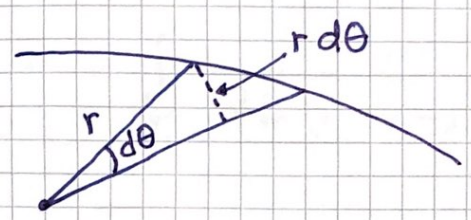
$$= L(r, \dot{r}, \dot{\theta})$$ [cf Ex 1b, p 11]

L independent of θ (θ is cyclic coordinate)

$$\Rightarrow \quad p_\theta = \partial L / \partial \dot\theta = m r^2 \dot\theta \quad \text{is conserved}$$

$$= \ell = \text{abs. value of the angular momentum}$$

Kepler's 2. law:



$$dA = \frac{1}{2} r \cdot r\, d\theta$$

$$\Rightarrow \dot A = \frac{1}{2} r^2 \dot\theta$$

$$= \frac{1}{2m} p_\theta = \text{constant}$$

The radius r sweeps across a constant area pr unit time

Eqn. of motion for r:

$$\frac{\partial L}{\partial r} = m r \dot\theta^2 - \frac{\partial V}{\partial r} \; ; \quad \frac{\partial L}{\partial \dot r} = m \dot r \; ; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot r} = m \ddot r$$

$$= \frac{\ell^2}{m r^3} + f(r) \qquad (\text{using } \dot\theta = \ell / m r^2)$$

$$\Rightarrow \quad m \ddot r - \frac{\ell^2}{m r^3} = f(r)$$

$$f(r) = -\partial V / \partial r = \text{force in radial direction } \hat r$$

Conservative system $\Rightarrow$ also total energy E is conserved,

$$E = T + V = \frac{1}{2} m \left(\dot r^2 + \overbrace{r^2 \dot\theta^2}^{\ell^2 / m^2 r^2} \right) + V(r)$$

Two 2. order eqns. have been transformed into two 1. order eqns. using conservation of ℓ and E.

Solution for $r(t)$:

$$\frac{1}{2} m \dot{r}^2 + \frac{\ell^2}{2mr^2} + V = E$$

$$\Rightarrow \frac{dr}{dt} = \pm \sqrt{\frac{2}{m}\left(E - V - \ell^2/2mr^2\right)}$$

Assume $r(t=0) = r_0$ = minimum radius:

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \ell^2/2mr^2\right)}} = t(r;\ E, \ell, r_0)$$

$\Rightarrow r(t)$ by inverting $t(r)$

Solution for $\theta(t)$:

$$\dot{\theta} = \ell / m r^2 \Rightarrow d\theta = \ell\, dt / m r^2(t)$$

$$\Rightarrow \theta(t) = \ell \int_0^t \frac{dt}{m r^2(t)} + \theta_0 \quad ; \quad \theta_0 = \theta(t=0)$$

Integration constants: E, ℓ, r_0, θ_0 (4, as expected)

May choose $r_0, \theta_0, \dot{r}_0, \dot{\theta}_0$ in classical mechanics, but better with E and ℓ e.g. when comparing classical with quantum mechanics since initial values for all four would violate Heisenberg's uncertainty relations.

3.3 Equivalent onedimensional problem

General insight from equation for r viewed as a onedimensional problem. We had (p. 38):

$$m\ddot{r} = f(r) + \frac{\ell^2}{mr^3} = f^*(r)$$

i.e. N2 for 1D problem with mass m and force

$$f^* = f + \ell^2/mr^3$$

Centrifugal force:

$$\ell^2/mr^3 = (mr^2\dot{\theta})^2/mr^3 = mr\dot{\theta}^2 = m v_\theta^2/r$$

Energy considerations; we had (p. 39):

$$E = \frac{1}{2} m\dot{r}^2 + \ell^2/2mr^2 + V(r)$$

i.e. 1D problem with potential

$$V^*(r) = V(r) + \ell^2/2mr^2$$

and corresponding force

$$f^* = -\partial V^*/\partial r = -\partial V/\partial r + \ell^2/mr^3$$

$$= f + \ell^2/mr^3$$

as above.

Ex 1. Attractive $\frac{1}{r^2}$ force

$$f(r) = -k/r^2 \quad \Rightarrow \quad V(r) = -k/r$$

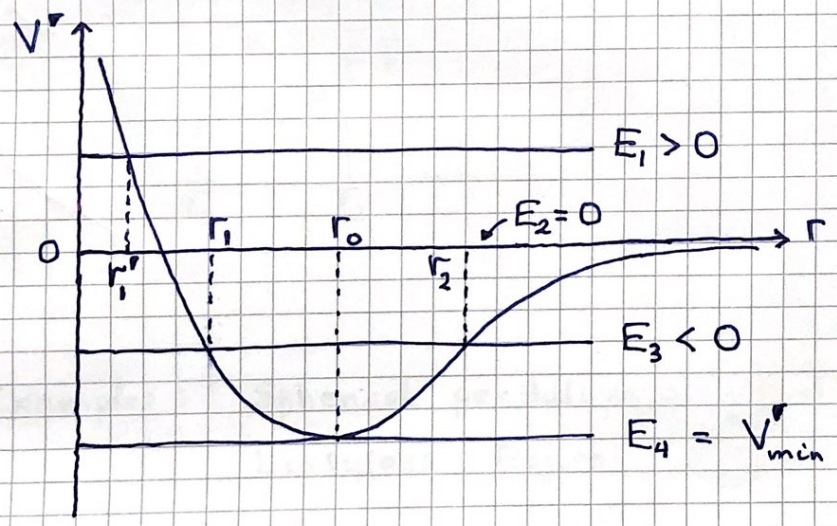
$$\Rightarrow V'(r) = -\frac{k}{r} + \frac{\ell^2}{2mr^2} \longrightarrow \begin{cases} +\infty & ; \ r \to 0 \\ -0 & ; \ r \to \infty \end{cases}$$

Centrifugal barrier : $\ell^2/2mr^2$

[Cf. Coulomb potential in QM, with quantized $\vec{L}^2$,

$$V_{eff}^{\ell}(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{\ell(\ell+1)\hbar^2}{2mr^2} \quad ; \quad \ell = 0, 1, 2, \ldots \quad]$$

$$E = \frac{1}{2}m\dot{r}^2 + V'(r) \geq V'(r)$$



$E_1 > 0$: unbounded motion, $r \geq r_1'$ (hyperbola)

$E_2 = 0$: ——— " ——— (parabola)

$E_3 < 0$: bounded motion, $r_1 \leq r \leq r_2$ (ellipse)

$E_4 = V'_{min}$: ——— " ——— , $r = r_0$ (circle)

As 1D problem : m at rest, force f balances centrifugal force

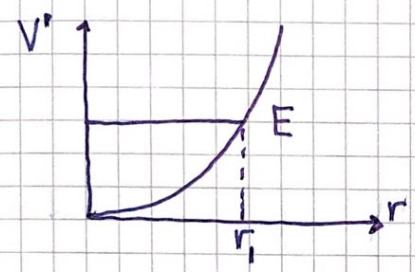
More about the Kepler problem and orbits in 3.7.

Ex. 2: Isotropic harmonic oscillator

$$f(r) = kr \quad ; \quad V(r) = \frac{1}{2}kr^2 \quad ; \quad V'(r) = \frac{1}{2}kr^2 + \frac{\ell^2}{2mr^2}$$

$\ell = 0$: (~~Null dreieimpuls~~) Zero angular momentum

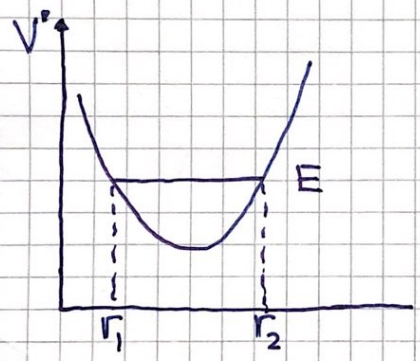
$\Rightarrow \vec{r} \parallel \vec{v} \quad \Rightarrow$ m moves along a straight line



$V' = V = \frac{1}{2}kr^2$

Bounded harmonic motion with $r \leq r_1$

$\ell > 0$:



$$V' = \frac{1}{2}kr^2 + \frac{\ell^2}{2mr^2}$$

Bounded motion with

$$r_1 \leq r \leq r_2$$

Examples: Spherical pendulum

Lissajous figures

(see assignments)

3.4 The virial theorem

Assume system of point masses m_i in positions $\vec{r}_i$, influenced by forces $\vec{F}_i$. Consider the function

$$G = \sum_i \vec{p}_i \cdot \vec{r}_i \quad \text{and} \quad \dot{G} = \sum_i \dot{\vec{r}}_i \cdot \vec{p}_i + \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i$$

1. sum: $\sum_i \dot{\vec{r}}_i \cdot m_i \dot{\vec{r}}_i = \sum_i m_i v_i^2 = 2T$

2. sum: $\sum_i \dot{\vec{p}}_i \cdot \vec{r}_i \overset{N2}{=} \sum_i \vec{F}_i \cdot \vec{r}_i$

$$\Rightarrow \quad \dot{G} = 2T + \sum_i \vec{F}_i \cdot \vec{r}_i$$

Take time average over time interval τ:

$$\overline{\dot{G}} = \frac{1}{\tau} \int_0^\tau \frac{dG}{dt}\, dt = \frac{1}{\tau}\left[G(\tau) - G(0)\right]$$

Assume long time interval: $\lim_{\tau \to \infty} \overline{\dot{G}} = 0$
(or periodic motion with period τ)

$$\Rightarrow \quad \overline{T} = -\frac{1}{2} \overline{\sum_i \vec{F}_i \cdot \vec{r}_i} \qquad \underline{\text{Virial theorem}}$$

Ex: Single particle in conservative central field

$$\overline{T} = \frac{1}{2} \overline{(\partial V/\partial r) \cdot r} \qquad \text{since} \quad \vec{F} = -\hat{r}\, \frac{\partial V}{\partial r}$$

Assume $V(r) = a\, r^{n+1}$; then $r\, \partial V/\partial r = (n+1)\, V$

$$\Rightarrow \overline{T} = \frac{n+1}{2} \overline{V}$$

Harmonic oscillator: $n = 1 \Rightarrow \overline{T} = \overline{V}$

Gravitational field: $n = -2 \Rightarrow \overline{T} = -\frac{1}{2} \overline{V}$

3.7 The Kepler problem

We will find the analytical solution $r(\theta)$ when $f(r) = -k/r^2$ and $V(r) = -k/r$ (Ex 1, p. 41). We had:

$$\frac{1}{2} m \dot{r}^2 + \frac{\ell^2}{2mr^2} + V = E \quad \text{and} \quad \dot{\theta} = \frac{\ell}{mr^2}$$

$$\Rightarrow \quad dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \ell^2/2mr^2\right)}} \quad \text{and} \quad dt = \frac{mr^2}{\ell} d\theta$$

$$\Rightarrow \quad d\theta = \frac{\ell \, dr}{mr^2 \sqrt{\frac{2}{m}\left(E - V - \ell^2/2mr^2\right)}}$$

$$\Rightarrow \quad \theta = \int_{r_0}^{r} \frac{dr}{r^2 \sqrt{\frac{2mE}{\ell^2} + \frac{2mk}{r\ell^2} - \frac{1}{r^2}}} + \theta_0$$

Choose $\theta_0 = \theta(r_0) = 0$ in <u>perihel</u>, where the particle is nearest the origin.

Substitute $u = 1/r \Rightarrow du = -dr/r^2$:

$$\theta = -\int_{u_0}^{u} du \left(\frac{2mE}{\ell^2} + \frac{2mku}{\ell^2} - u^2 \right)^{-1/2}$$

"Standard integral":

$$\int dx \left(\alpha + \beta x - x^2\right)^{-1/2} = \arccos \frac{2x - \beta}{\sqrt{4\alpha + \beta^2}} = \arccos \frac{\frac{2x}{\beta} - 1}{\sqrt{1 + \frac{4\alpha}{\beta^2}}}$$

$$\Rightarrow \theta = -\arccos \frac{\ell^2/mkr - 1}{\sqrt{1 + 2E\ell^2/mk^2}} = -\arccos \frac{p/r - 1}{\varepsilon}$$

$$\Rightarrow \cos\theta = \cos(-\theta) = \frac{p/r - 1}{\varepsilon} \quad \Rightarrow \quad \underline{r = \frac{p}{1 + \varepsilon \cos\theta}}$$

Here, we have introduced

the eccentricity : $\varepsilon = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$

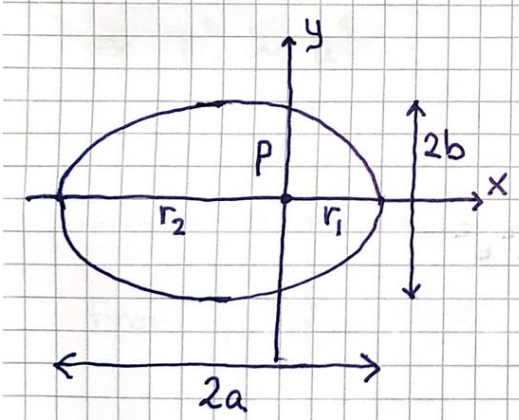
the "parameter of the path" : $p = \ell^2 / mk$

The equation $r = p/(1+\varepsilon\cos\theta)$ describes a conic section with focal point (focus) in the origin. Notice that $\theta = 0$ corresponds to $r_{min} = p/(1+\varepsilon)$; perihel.

Classification of various types of paths:

$\varepsilon > 1$:	hyperbola	$(E > 0)$
$\varepsilon = 1$:	parabola	$(E = 0)$
$\varepsilon < 1$:	ellipse	$(E < 0)$
$\varepsilon = 0$:	circle	$(E = -mk^2/2\ell^2)$

Elliptic orbit:



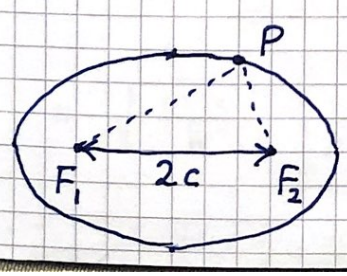
semi-major axis : a

semi-minor axis : b

$$2a = r_1 + r_2 = \frac{p}{1+\varepsilon} + \frac{p}{1-\varepsilon} = \frac{2p}{1-\varepsilon^2}$$

$$\Rightarrow a = \frac{p}{1-\varepsilon^2} = -\frac{k}{2E} = \frac{k}{2|E|} \quad (E<0)$$

Focal points:



$$PF_1 + PF_2 = 2a$$

$$\varepsilon = c/a$$

$$a^2 = b^2 + c^2$$

$$\Rightarrow b^2 = a^2 - c^2 = a^2(1-\varepsilon^2) = \left[\frac{p}{1-\varepsilon^2}\right]^2 (1-\varepsilon^2)$$
$$= \frac{p^2}{1-\varepsilon^2}$$

$$\Rightarrow b = \frac{p}{\sqrt{1-\varepsilon^2}} = \frac{\ell^2/mk}{\sqrt{-2E\ell^2/mk^2}} = \frac{\ell}{\sqrt{-2mE}} = \frac{\ell}{\sqrt{2m|E|}}$$

Period :

From p. 38 : $\ell = mr^2\dot{\theta}$, $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{\ell}{2m}$

$$\Rightarrow A = \int dA = \int_0^T \frac{dA}{dt}\,dt = \frac{\ell}{2m}\cdot T$$

Ellipse : $A = \pi ab$

$$a = k/2|E| \Rightarrow 1/\sqrt{2|E|} = \sqrt{a/k} \Rightarrow \frac{b}{\ell} = \sqrt{\frac{a}{m\cdot k}}$$

$$\Rightarrow T = 2\pi m\cdot a\cdot\frac{b}{\ell} = 2\pi\sqrt{\frac{m}{k}}\cdot a^{3/2}$$

$\Rightarrow T^2 \sim a^3$; Kepler's 3. law

The motion in time :

From p. 39, with $V = -k/r$:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}\left(E + k/r - \ell^2/2mr^2\right)}} = \sqrt{\frac{m}{2|E|}}\int \frac{r\,dr}{\sqrt{-r^2 + \frac{kr}{|E|} - \frac{\ell^2}{2m|E|}}}$$

$$= \sqrt{\frac{m\cdot a}{k}}\int \frac{r\,dr}{\sqrt{-r^2 + 2ar - b^2}}$$

Since $-b^2 = -a^2 + a^2\varepsilon^2$, we have

$$-r^2 + 2ar - b^2 = -(r-a)^2 + a^2\varepsilon^2$$

Substitute $r = a - a\varepsilon\cos\xi$

Then: $r \cdot dr = (a - a\varepsilon\cos\xi)\cdot a\varepsilon\sin\xi \cdot d\xi$

$$\sqrt{-r^2+2ar-b^2} = \sqrt{a^2\varepsilon^2 - a^2\varepsilon^2\cos^2\xi} = a\varepsilon\sin\xi$$

$$\Rightarrow t = \sqrt{\frac{ma^3}{k}}\int(1-\varepsilon\cos\xi)\,d\xi = \sqrt{\frac{ma^3}{k}}(\xi - \varepsilon\sin\xi)$$

where we chose $t=0$ for $\xi = 0$. Then, at $t=0$,

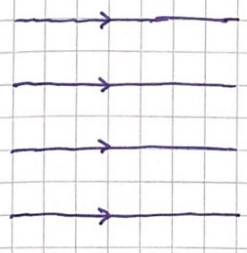
$$r = a(1-\varepsilon) = r_{min} \quad \text{(perihel)}$$

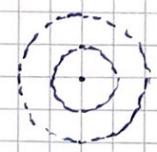
since $r_{min} = p/(1+\varepsilon)$, $a = p/(1-\varepsilon^2)$ $\Rightarrow \frac{r_{min}}{a} = 1-\varepsilon$

Comments:

- $\xi = 2\pi \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}\,a^{3/2}$, as before
- $\cos\theta = (\cos\xi - \varepsilon)/(1-\varepsilon\cos\xi)$ $\Rightarrow \theta = 0$ at $t=0$ (OK)
  (See textbook for details)

3.10 Scattering in a central force field





Central forcefield, attractive or repulsive.

$$f(r) = -\frac{\partial V}{\partial r}$$

$$\lim_{r\to\infty} f(r) = 0$$

Uniform flow of particles with equal mass m and energy E.

I = intensity = # particles pr unit time, and unit area $\perp$ particle flow

$\sigma(\Omega)$ = differential scattering cross section

$\sigma(\Omega)\, d\Omega$ = (# particles scattered into solid angle $d\Omega$ pr unit time) / I

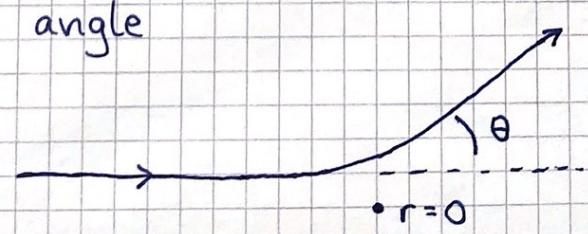
$$d\Omega = \sin\theta \, d\theta \, d\varphi$$

$$[\sigma] = s^{-1} / s^{-1} m^{-2} = m^2$$

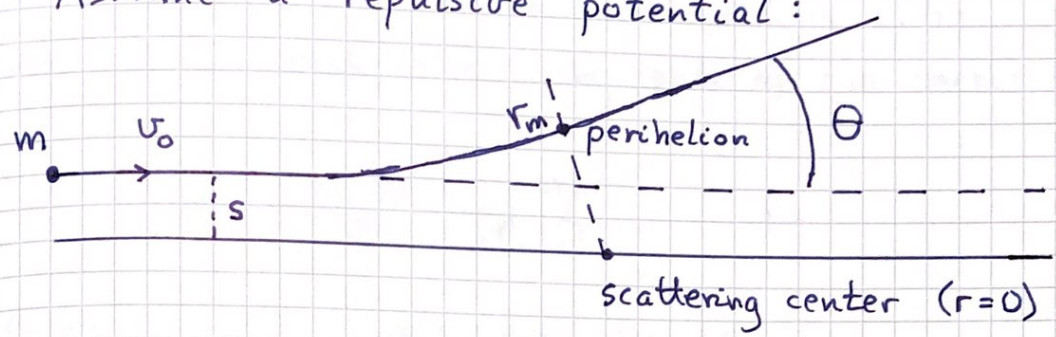
Central force $\Rightarrow$ Symmetry around incoming axis

$\Rightarrow d\Omega = 2\pi \sin\theta \, d\theta$

θ = scattering angle



Assume a repulsive potential:



s = impact parameter

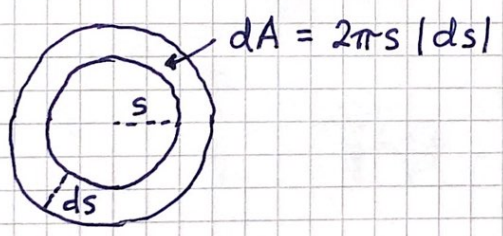
$$\ell = \lim_{r\to\infty} |\vec{r}\times\vec{p}| = m v_0 s = m\cdot\sqrt{\frac{2E}{m}}\cdot s = s\cdot\sqrt{2mE}$$

$$\theta = \theta(s) \; ; \quad \theta(s=0) = \pi \; ; \quad \theta(s\to\infty) = 0$$

θ decreases steadily with increasing s (for repulsive V)

$\Rightarrow$ # particles in between s and s+ds

= # particles out between θ and $\theta + d\theta$



$$\Rightarrow \quad I\cdot \underbrace{2\pi s\,|ds|}_{dA} = I\cdot\sigma(\theta)\cdot \underbrace{2\pi\sin\theta\cdot|d\theta|}_{d\Omega}$$

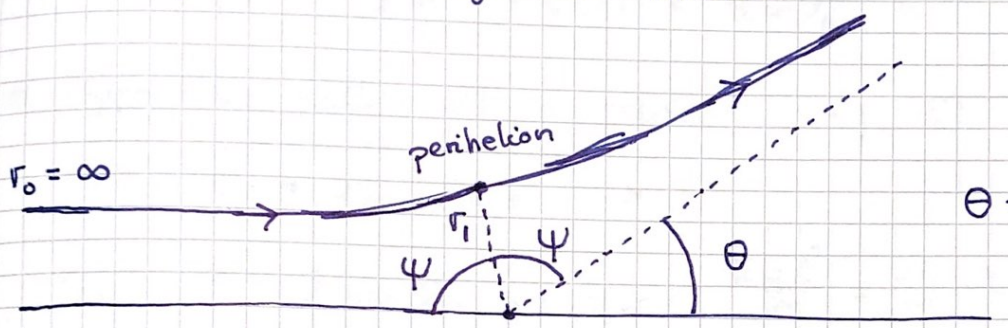
[Here, $ds/d\theta < 0$, hence $|ds|$ and $|d\theta|$]

$$\Rightarrow \quad \sigma(\theta) = \frac{s}{\sin\theta}\cdot\left|\frac{ds}{d\theta}\right|$$

We need $s(\theta)$. Start from integral on p. 44 with β as angle along the path of the particle:

$$\beta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}} + \beta_0$$

Geometry and integration limits:



$$\theta + 2\psi = \pi$$

$$r_0 = \infty \quad \Rightarrow \quad \beta_0 = \pi$$

$$r = r_1 \quad \Rightarrow \quad \beta = \pi - \psi$$

$$\Rightarrow \quad \pi - \psi = \int_{\infty}^{r_1} \frac{dr}{r^2\sqrt{\cdots\cdots}} + \pi$$

$$\Rightarrow \quad \psi = \int_{r_1}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}}$$

Now, introduce s:

$$\ell = s\sqrt{2mE} \quad \Rightarrow \quad \frac{2mE}{\ell^2} = \frac{1}{s^2} \quad ; \quad \frac{2mV}{\ell^2} = \frac{V}{E \cdot s^2}$$

$$\Rightarrow \psi = \int_{r_1}^{\infty} \frac{s\,dr}{r^2\sqrt{1 - \frac{V}{E} - s^2/r^2}}$$

or with $u = 1/r$, $du = -dr/r^2$, $u_1 = 1/r_1$; $\theta = \pi - 2\psi$:

$$\theta(s) = \pi - 2\int_0^{u_1} \frac{s \cdot du}{\sqrt{1 - V/E - s^2u^2}}$$

Ex. with analytical solution:

$m \bullet\!\!\longrightarrow v_0$ $\qquad$ $M \gg m$ (Scattering center and mass center)

$q' = Z'e$ $\qquad$ $q = Ze$

Coulomb potential: $V = -\frac{k}{r}$; $k = -ZZ'e^2/4\pi\varepsilon_0 < 0$

$$E = T + V = \tfrac{1}{2}mv^2 + V(r) = \tfrac{1}{2}mv_0^2 > 0$$

$\Rightarrow$ Hyperbolic path with eccentricity (p. 45)

$$\varepsilon = \sqrt{1 + 2E\ell^2/mk^2} > 1$$

Since $\ell^2 = s^2 \cdot 2mE$, we have $\frac{2E\ell^2}{m} = \frac{2E}{m} \cdot s^2 \cdot 2mE = 4E^2s^2$

$$\Rightarrow \quad \varepsilon = \sqrt{1 + 4E^2s^2/k^2}$$

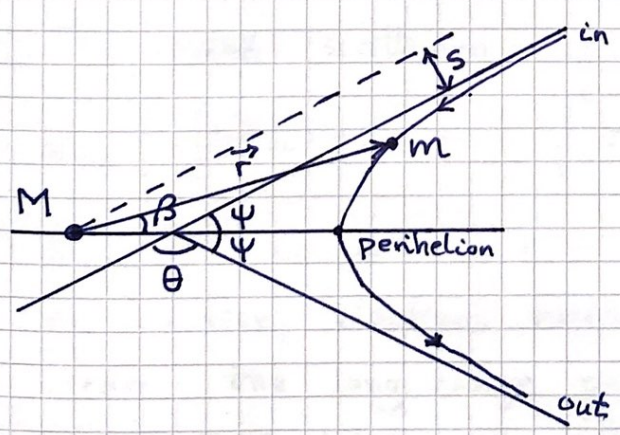
Parameter of the path: $p = \ell^2/mk < 0$

The derivation on p. 44 now yields the result

$$\cos\beta = \frac{1 - p/r}{\varepsilon} = \frac{1 + |p|/r}{\varepsilon}$$

or $|p|/r = \varepsilon\cos\beta - 1$ or $\underline{r = \frac{|p|}{\varepsilon\cos\beta - 1}}$

for the conic section.



Asymptotes:

$r \to \infty$

$\Rightarrow \quad \beta \to (\pm)\,\psi$

$\Rightarrow \quad \varepsilon\cos\psi = 1$

$\Rightarrow \quad \cos\psi = 1/\varepsilon$

Scattering angle: $\theta = \pi - 2\psi \Rightarrow \psi = \frac{\pi}{2} - \frac{\theta}{2}$

$$\Rightarrow \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \sin\frac{\theta}{2} = \frac{1}{\varepsilon}$$

$$\Rightarrow \varepsilon^2 - 1 = \frac{1}{\sin^2\theta/2} - \frac{\sin^2\theta/2}{\sin^2\theta/2} = \frac{\cos^2\theta/2}{\sin^2\theta/2} = \underline{\cot^2\theta/2}$$

$\parallel$ (p.51)

$\underline{4E^2 s^2/k^2}$

$$\Rightarrow s(\theta) = \frac{|k|}{2E}\cot\frac{\theta}{2}$$

[sign check: $0 < \theta < \pi \Rightarrow 0 < \frac{\theta}{2} < \frac{\pi}{2} \Rightarrow \cot\frac{\theta}{2} > 0$]

$$\Rightarrow \left|\frac{ds}{d\theta}\right| = \frac{|k|}{4E}\cdot\frac{1}{\sin^2(\theta/2)}$$

$$\Rightarrow \sigma(\theta) = \frac{s}{\sin\theta}\cdot\left|\frac{ds}{d\theta}\right| = \frac{k^2}{8E^2}\cdot\frac{\cot\frac{\theta}{2}}{\sin\theta\cdot\sin^2\theta/2}$$

$\cot\theta/2 = \cos(\theta/2)/\sin(\theta/2)$

$\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$

$$\Rightarrow \underline{\sigma(\theta) = \left(\frac{k}{4E}\right)^2\cdot\frac{1}{\sin^4(\theta/2)}}$$

The Rutherford scattering cross section

Comments:

- Surprisingly, nonrelativistic quantum mechanics gives exactly the same result.
- The <u>total</u> scattering cross section <u>diverges</u>:

$$\sigma_{TOT} = \int\sigma(\Omega)\,d\Omega = 2\pi\int_0^\pi \sigma(\theta)\sin\theta\,d\theta \rightarrow \infty$$

I.e., the Coulomb force is long range.
In practice, electrons surrounding the nucleus "screen" the long range potential and results in a finite total cross section.

4. Rigid body kinematics

4.1 Independent coordinates

N particles $\Rightarrow$ 3N degrees of freedom

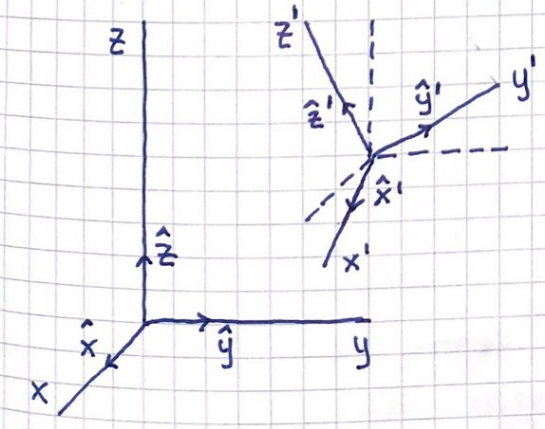
Holonomic constraints : $r_{ij} = c_{ij}$ [$\frac{1}{2}N(N-1)$ constraints]

$\Rightarrow$ Only 6 degrees of freedom :

- 3 coords. to specify a reference point in the rigid body (e.g. CM)
- 3 coords. (angles) to specify orientation of the rigid body relative to the chosen reference frame

Alternatively :
6 independent movements, translation along 3 axes and rotation around 3 axes.

Let x, y, z be axes in a fixed (external) coord. system, and x', y', z' axes in a coord. system fixed in the rigid body :



3 coords. specify the origin in the $(x'\,y'\,z')$ system.

$\Rightarrow$ Need 3 more to specify the directions of x', y', z' relative to x, y, z

Direction cosines :

$$\alpha_1 = \cos(\hat{x}', \hat{x}) = \hat{x}' \cdot \hat{x}$$
$$\alpha_2 = \cos(\hat{x}', \hat{y}) = \hat{x}' \cdot \hat{y}$$
$$\alpha_3 = \cos(\hat{x}', \hat{z}) = \hat{x}' \cdot \hat{z}$$

Similarly with β for $\hat{y}'$ and γ for $\hat{z}'$.
The direction cosines are the components of $\hat{x}', \hat{y}', \hat{z}'$ along the external axes $\hat{x}, \hat{y}, \hat{z}$:

$$\hat{x}' = \alpha_1 \hat{x} + \alpha_2 \hat{y} + \alpha_3 \hat{z}$$
$$\hat{y}' = \beta_1 \hat{x} + \beta_2 \hat{y} + \beta_3 \hat{z}$$
$$\hat{z}' = \gamma_1 \hat{x} + \gamma_2 \hat{y} + \gamma_3 \hat{z}$$

And the other way around :

$$\hat{x} = \alpha_1 \hat{x}' + \beta_1 \hat{y}' + \gamma_1 \hat{z}'$$
$$\hat{y} = \alpha_2 \hat{x}' + \beta_2 \hat{y}' + \gamma_2 \hat{z}'$$
$$\hat{z} = \alpha_3 \hat{x}' + \beta_3 \hat{y}' + \gamma_3 \hat{z}'$$

6 constraints reduce 9 direction cosines to 3 independent coordinates :

$$\hat{x} \cdot \hat{x} = (\alpha_1 \hat{x}' + \beta_1 \hat{y}' + \gamma_1 \hat{z}')^2 = \alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1$$

$$\hat{x} \cdot \hat{y} = \alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2 = 0$$

etc.

$$\Rightarrow \quad \alpha_i \alpha_j + \beta_i \beta_j + \gamma_i \gamma_j = \delta_{ij}$$

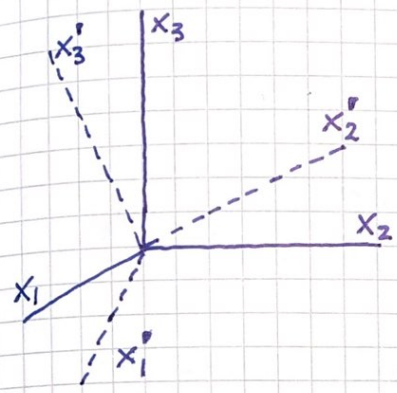
Soon : Euler angles, 3 independent functions of the direction cosines

4.2 Orthogonal transformations

Focus on rotation.

Assume common origin in external and body system.

Notation: $x, y, z \rightarrow x_1, x_2, x_3$ (more convenient)



Coord. transformations:

$$x_1' = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$
$$x_2' = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$
$$x_3' = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3$$

i.e., a linear transf.

General notation:

$$x_1' = a_{11} x_1 + a_{12} x_2 + a_{13} x_3$$
$$x_2' = a_{21} x_1 + a_{22} x_2 + a_{23} x_3$$
$$x_3' = a_{31} x_1 + a_{32} x_2 + a_{33} x_3$$

With summation convention:

$$x_i' = a_{ij} x_j \quad ; \quad i = 1, 2, 3 \qquad (1)$$

A rotation leaves $r = |\vec{r}|$ unchanged:

$$x_i' x_i' = x_i x_i \quad \Rightarrow \quad a_{ij} x_j a_{ik} x_k = x_i x_i$$

$$\Rightarrow \quad a_{ij} a_{ik} = \delta_{jk} \qquad (2)$$

i.e., the 6 constraints on p. 54.

When (2) are satisfied, (1) is called an orthogonal transformation, and (2) is the orthogonality condition.

Transformation matrix:

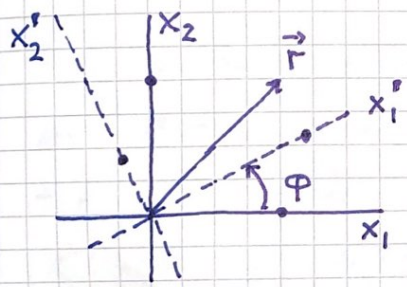
$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$ with matrix elements a_{ij}

Ex: Rotation in 2D

$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad ; \quad a_{ij}\, a_{ik} = \delta_{jk}$$

4 matrix elements, 3 orthog. conditions

$\Rightarrow$ 1 independent coordinate, the rotation angle φ, OK!



From the figure (check yourself):

$$x_1' = x_1 \cos\varphi + x_2 \sin\varphi$$
$$x_2' = -x_1 \sin\varphi + x_2 \cos\varphi$$
$$\Rightarrow a_{11} = a_{22} = \cos\varphi \; ; \; a_{12} = -a_{21} = \sin\varphi$$

$$\Rightarrow \mathbb{A} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

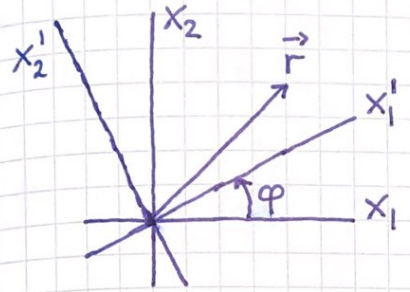
Orthog. conditions:

$$\left.\begin{array}{lll} j=k=1: & a_{11}a_{11} + a_{21}a_{21} = 1 & \Rightarrow \cos^2\varphi + \sin^2\varphi = 1 \\ j=k=2: & a_{12}a_{12} + a_{22}a_{22} = 1 & \Rightarrow \sin^2\varphi + \cos^2\varphi = 1 \\ j\neq k: & a_{11}a_{12} + a_{21}a_{22} = 0 & \Rightarrow \cos\varphi\sin\varphi - \sin\varphi\cos\varphi = 0 \end{array}\right\} \text{OK!}$$

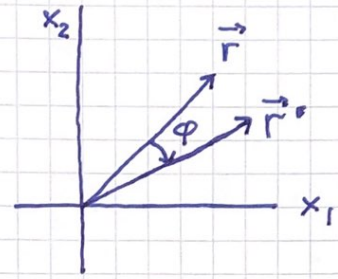
Expressed in 3D, rotation around z axis:

$$\mathbb{A} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Two interpretations of the transformation $\vec{r}\,' = \mathbb{A}\vec{r}$:



Passive



Active

Passive point of view:

Operator $\mathbb{A}$ rotates the coord. system, here counterclockwise. Vector $\vec{r}$ is fixed, $\vec{r}\,'$ expresses components of $\vec{r}$ in rotated coord. system.

Active point of view:

$\mathbb{A}$ rotates the vector $\vec{r}$, here clockwise, while the coord. system is fixed. $\vec{r}\,'$ contains components x_1' and x_2' after the rotation.

4.3 Formal properties of the transformation matrix

(see also MA1201, MA1202, TMA4115)

Notation: $\vec{r} \to \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

- Two successive orthogonal transf. $\mathbb{B}$ followed by $\mathbb{A}$ equals a single orthog. transf. $\mathbb{C} = \mathbb{A}\mathbb{B}$:

$$x_k' = b_{kj} x_j \; ; \; x_i'' = a_{ik} x_k' = a_{ik} b_{kj} x_j \equiv c_{ij} x_j$$

$$\Rightarrow c_{ij} = a_{ik} b_{kj} \qquad \text{(remember summation convention)}$$

- In general, $\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}$ but $(\mathbb{A}\mathbb{B})\mathbb{C} = \mathbb{A}(\mathbb{B}\mathbb{C})$, i.e. orthog. transf. are associative but not commutative.

- The inverse transf. $\mathbb{A}^{-1}$, with matrix elements a^{-1}_{ij} (NB: $a^{-1}_{ij} \neq 1/a_{ij}$), brings $\vec{x}'$ back to $\vec{x}$:

$$x_i = a^{-1}_{ij} x_j' \quad \Rightarrow x_k' = a_{ki} x_i = a_{ki} a^{-1}_{ij} x_j'$$

$$\Rightarrow \underbrace{a_{ki} a^{-1}_{ij}}_{(\mathbb{A}\mathbb{A}^{-1})_{kj}} = \underbrace{\delta_{kj}}_{\mathbb{1}_{kj}} \quad \Rightarrow \mathbb{A}\mathbb{A}^{-1} = \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Also: $x_i = a^{-1}_{ij} x_j' = a^{-1}_{ij} a_{jk} x_k \Rightarrow a^{-1}_{ij} a_{jk} = \delta_{ik}$

$\Rightarrow \mathbb{A}^{-1}\mathbb{A} = \mathbb{1} \quad \Rightarrow \mathbb{A}$ and $\mathbb{A}^{-1}$ commute

- For an orthogonal matrix, $\mathbb{A}^{-1} = \tilde{\mathbb{A}}$ = transposed of $\mathbb{A}$

Proof: $a_{k\ell} a_{ki} a^{-1}_{ij} = \delta_{i\ell} a^{-1}_{ij} = a^{-1}_{\ell j}$ (using orthog. cond.) <u>and</u> $a_{k\ell} a_{ki} a^{-1}_{ij} = a_{k\ell} \delta_{kj} = a_{j\ell}$ (using $\mathbb{A}\mathbb{A}^{-1} = \mathbb{1}$). Hence, $a^{-1}_{\ell j} = a_{j\ell} = \tilde{a}_{\ell j}$, i.e., $\mathbb{A}^{-1} = \tilde{\mathbb{A}}$

- Alternative expressions for orthog. conditions:

$\tilde{\mathbb{A}}\mathbb{A} = \mathbb{1} \Rightarrow \tilde{a}_{ij}\, a_{jk} = \delta_{ik} \Rightarrow a_{ji}\, a_{jk} = \delta_{ik}$

$\mathbb{A}\tilde{\mathbb{A}} = \mathbb{1} \Rightarrow a_{ij}\, \tilde{a}_{jk} = \delta_{ik} \Rightarrow a_{ij}\, a_{kj} = \delta_{ik}$

I.e., may sum over 1. or 2. index

- Determinants: $|\mathbb{A}\mathbb{B}| = |\mathbb{A}| \cdot |\mathbb{B}|$ (in general)

$\Rightarrow |\tilde{\mathbb{A}}| \cdot |\mathbb{A}| = 1$ and since $|\tilde{\mathbb{A}}| = |\mathbb{A}|$ (—"—)

we find $|\mathbb{A}|^2 = 1$, i.e., $|\mathbb{A}| = \pm 1$

- The orthog. transf. must be realizable for a rigid body, hence $|\mathbb{A}| = +1$

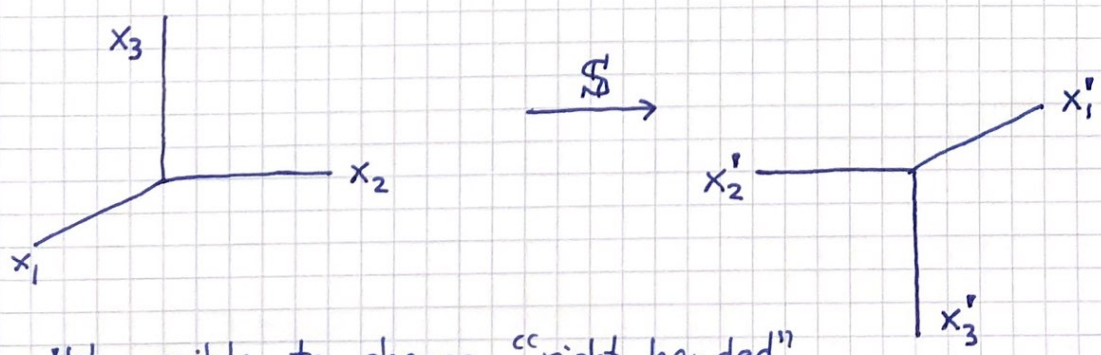
Argument 1: $|\mathbb{A}|$ must evolve continuously from the unit matrix $\Rightarrow |\mathbb{A}| = |\mathbb{1}| = +1$.

Argument 2, by an example:

Consider the transf. $\vec{x}' = \mathbb{S}\,\vec{x}$ with

$$\mathbb{S} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \Rightarrow |\mathbb{S}| = -1$$

i.e. reflection of all 3 axes, $x_1' = -x_1$ etc.



Not possible to change "right handed" rigid body into a "left handed" one by rotations and translation

4.4 The Euler angles

We need 3 independent coords. that specify the rigid body orientation, such that $|\mathbb{A}| = +1$. Common choice is the Euler angles, which are 3 successive rotation angles. $\mathbb{A} = \mathbb{B}\,\mathbb{C}\,\mathbb{D}$

1. Rot. $\varphi > 0$ around z axis, $xyz \to \xi\eta\zeta$

$$\boldsymbol{\xi} = \mathbb{D}\,\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{\xi} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

$$\mathbb{D} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Rot. $\theta > 0$ around ξ axis, $\xi\eta\zeta \to \xi'\eta'\zeta'$

$$\boldsymbol{\xi}' = \mathbb{C}\,\boldsymbol{\xi}, \quad \boldsymbol{\xi}' = \begin{pmatrix} \xi' \\ \eta' \\ \zeta' \end{pmatrix}$$

$$\mathbb{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

3. Rot. $\psi > 0$ around ζ' axis, $\xi'\eta'\zeta' \to x'y'z'$

$$\mathbf{x}' = \mathbb{B}\,\boldsymbol{\xi}', \quad \mathbf{x}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\mathbb{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{x}' = \mathbb{A}\,\mathbb{x} = \mathbb{B}\,\mathbb{C}\,\mathbb{D}\,\mathbb{x}$$

$$\Rightarrow \quad a_{11} = \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi$$

$$a_{12} = \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi$$

$$a_{13} = \sin\psi\sin\theta$$

$$a_{21} = -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi$$

$$a_{22} = -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi$$

$$a_{23} = \cos\psi\sin\theta$$

$$a_{31} = \sin\theta\sin\varphi$$

$$a_{32} = -\sin\theta\cos\varphi$$

$$a_{33} = \cos\theta$$

Inverse transformation:

$$\mathbb{x} = \mathbb{A}^{-1}\,\mathbb{x}' = \tilde{\mathbb{A}}\,\mathbb{x}'$$

$$\tilde{a}_{ij} = a_{ji}$$

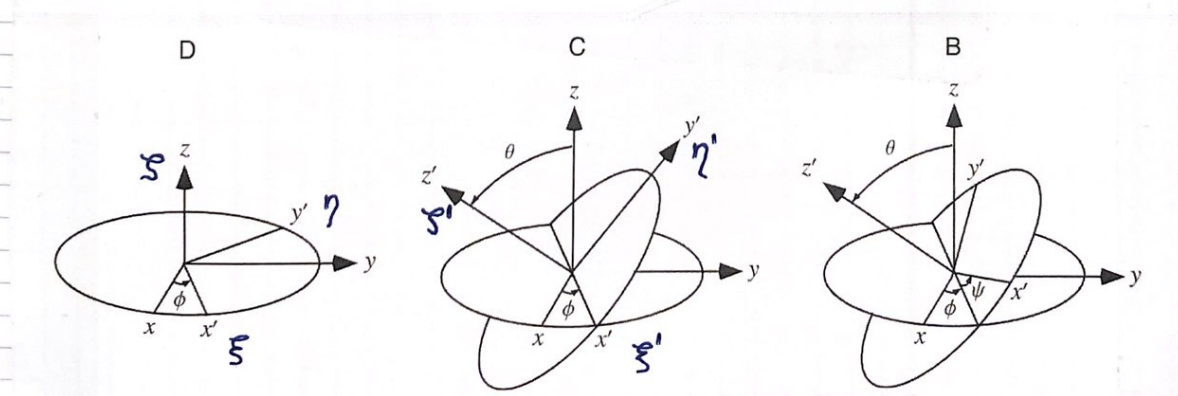


Fig: Wolfram MathWorld (E.W. Weisstein)

4.8 Infinitesimal rotations

In general, two <u>finite</u> rotations $\mathbb{A}$ and $\mathbb{B}$ do <u>not</u> commute.

Ex. 1 : $\mathbb{C}$ and $\mathbb{D}$ on p. 60.

Ex 2 : Matchbox rotated 90° around two orthogonal axes.

But <u>infinitesimal</u> transformations commute :

$$x_i' = x_i + \varepsilon_{ij} x_j = (\delta_{ij} + \varepsilon_{ij}) x_j \quad ; \quad \varepsilon_{ij} \ll 1$$

On matrix form : $\mathbb{X}' = (\mathbb{1} + \mathbb{E})\mathbb{X}$

Two successive inf. transf. :

$$\left.\begin{aligned} (\mathbb{1} + \mathbb{E}_1)(\mathbb{1} + \mathbb{E}_2) &= \mathbb{1} + \mathbb{E}_1 + \mathbb{E}_2 \\ (\mathbb{1} + \mathbb{E}_2)(\mathbb{1} + \mathbb{E}_1) &= \mathbb{1} + \mathbb{E}_1 + \mathbb{E}_2 \end{aligned}\right\} \text{commutative!}$$

Inverse inf. transf. :

$$\mathbb{A}^{-1} = \mathbb{1} - \mathbb{E} \qquad \text{since } \mathbb{A}\mathbb{A}^{-1} = (\mathbb{1} + \mathbb{E})(\mathbb{1} - \mathbb{E}) = \mathbb{1}$$

Orthogonality :

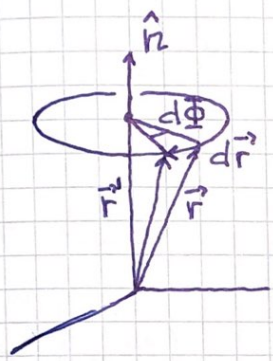
$$\tilde{\mathbb{A}} = \mathbb{1} + \tilde{\mathbb{E}} = \mathbb{A}^{-1} \Rightarrow \tilde{\mathbb{E}} = -\mathbb{E} \Rightarrow \tilde{\varepsilon}_{ij} = \varepsilon_{ji} = -\varepsilon_{ij}$$

i.e. $\mathbb{E}$ is antisymmetric 3×3 matrix on the form

$$\mathbb{E} = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbb{X}' - \mathbb{X} \equiv d\mathbb{X} = \mathbb{E}\,\mathbb{X} = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} dx_1 &= x_2\, d\Omega_3 - x_3\, d\Omega_2 \\ dx_2 &= x_3\, d\Omega_1 - x_1\, d\Omega_3 \\ dx_3 &= x_1\, d\Omega_2 - x_2\, d\Omega_1 \end{aligned} \qquad \Rightarrow d\vec{r} = \vec{r} \times d\vec{\Omega}$$



$d\vec{r} = \vec{r} \times d\vec{\Omega}$; $d\vec{\Omega} = \hat{n}\, d\Phi$

NB: $d\vec{\Omega}$ is <u>not</u> the differential of a finite vector $\vec{\Omega}$; $d\vec{\Omega}$ is a differential vector

$d\vec{r} = \vec{r}\,' - \vec{r}$ = change in the vector produced by an infinitesimal (clockwise) rotation of the vector

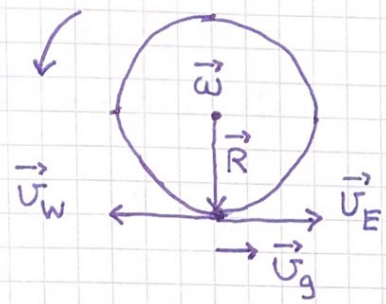
4.9 Rate of change of a vector

The change $d\vec{G}$ in an arbitrary (general) vector $\vec{G}$ is measured with different values in the coord. system fixed in the rigid body and the external coord. system:

$$(d\vec{G})_{body} \neq (d\vec{G})_{space}$$

The difference due to rotation of the body system in the space system is $\vec{G} \times d\vec{\Omega}$, and we fix the <u>sign</u> with an example.

A car drives east or west on the equator:



$\vec{\omega} = d\vec{\Omega}/dt$ out of the plane

$\vec{v}_g = \vec{\omega} \times \vec{R}$ = speed of ground measured in fixed external system

$\vec{v}_{E,W}$ = speed of car in rotating system

$$\Rightarrow \quad \vec{v}_c = \vec{v}_{E,W} + \vec{\omega} \times \vec{R}$$

= speed of car in external system

$$\Rightarrow \quad (d\vec{G})_{space} = (d\vec{G})_{body} + d\vec{\Omega} \times \vec{G}$$

$$\Rightarrow \quad \left(\frac{d\vec{G}}{dt}\right)_s = \left(\frac{d\vec{G}}{dt}\right)_b + \vec{\omega} \times \vec{G}$$

$\vec{\omega} = d\vec{\Omega}/dt$ = instantaneous angular velocity

Operator relation:

$$\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_b + \vec{\omega} \times$$

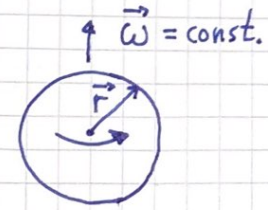
Ex: $\vec{G} = \vec{r} \quad \Rightarrow \quad \vec{v}_s = \vec{v}_b + \vec{\omega} \times \vec{r}$ (as above)

(For a more formal derivation of the operator relation, see Goldstein or handwritten compendium.)

4.10 Coriolis force and centrifugal force

s = approximate inertial system fixed relative to some nearby stars

b = system rotating with the earth



Assume common axes at a given instant.

$$\vec{G} = \vec{r} \quad \Rightarrow \quad \vec{v}_s = \vec{v}_b + \vec{\omega} \times \vec{r}$$

$$\vec{G} = \vec{v}_s \quad \Rightarrow \quad \left(\frac{d\vec{v}_s}{dt}\right)_s = \left(\frac{d\vec{v}_s}{dt}\right)_b + \vec{\omega} \times \vec{v}_s$$

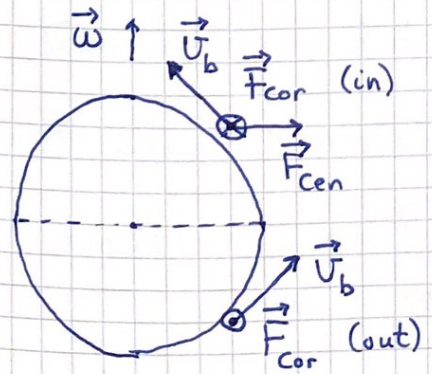
$$\left(\frac{d\vec{v}_s}{dt}\right)_b = \left(\frac{d\vec{v}_b}{dt}\right)_b + \left[\frac{d}{dt}(\vec{\omega} \times \vec{r})\right]_b = \vec{a}_b + \vec{\omega} \times \vec{v}_b$$

$$\vec{\omega} \times \vec{v}_s = \vec{\omega} \times \vec{v}_b + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\Rightarrow \vec{a}_s = \vec{a}_b + 2\,\vec{\omega} \times \vec{v}_b + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

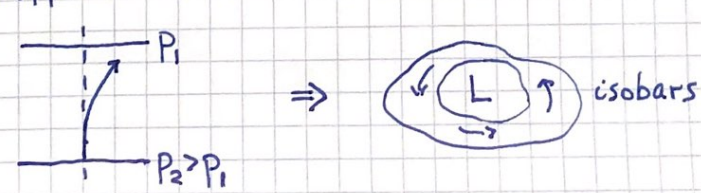
N2 in the inertial frame s, $\vec{F} = m\vec{a}_s$, can now be expressed as $\vec{F}_b = m\vec{a}_b$ with

$$\vec{F}_b = \vec{F} + \underbrace{2m\,\vec{v}_b \times \vec{\omega}}_{\text{Coriolis force}} \underbrace{- m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{Centrifugal force}}$$



$\vec{F}_{cor}$ ⇒ right deflection on northern hemisphere; left deflection on southern hemisphere

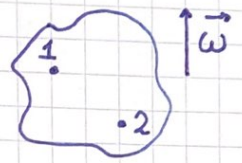
Affects wind direction:



5. Rigid body equations of motion

5.1 Angular momentum and kinetic energy

Rotating rigid body:



$$\vec{\omega}_1 = \vec{\omega}_2 = \vec{\omega}$$

Assume rotation around a fixed point = origo = CM.

Total angular momentum (around the fixed origin):

$$\vec{L} = m_i (\vec{r}_i \times \vec{v}_i) \qquad \text{[Sum convention]}$$

Rotation only $\Rightarrow$ $\vec{v}_i = \vec{\omega} \times \vec{r}_i$ Check this, using ε_{ijk}

$$\Rightarrow \vec{L} = m\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = m_i \left[\vec{\omega}\, r_i^2 - \vec{r}_i\, (\vec{r}_i \cdot \vec{\omega})\right]$$

Components of $\vec{L}$:

$$L_x = m_i \left[\omega_x r_i^2 - x_i (x_i \omega_x + y_i \omega_y + z_i \omega_z)\right]$$

$$= m_i \left[\omega_x (r_i^2 - x_i^2) - \omega_y x_i y_i - \omega_z x_i z_i\right]$$

On the general form (with j and k denoting components)

$$L_j = I_{jk}\, \omega_k \; ; \qquad I_{jk} = \text{(moment of) inertia tensor}$$

with elements $I_{xx} = m_i (r_i^2 - x_i^2)$, $I_{xy} = -m_i x_i y_i$ etc.

Continuous mass distribution: $m_i \to dm = \rho\, dV$; $\sum_i \to \int_V$

$$\Rightarrow I_{xx} = \int_V \rho(\vec{r})(r^2 - x^2)\, dV, \qquad I_{xy} = -\int_V \rho(\vec{r})\, xy\, dV$$

Compact: $I_{jk} = \int_V \rho(\vec{r}) \left(r^2 \delta_{jk} - x_j x_k\right) dV \qquad [xyz \to x_1 x_2 x_3]$

$$\vec{L} = \overleftrightarrow{I}\, \vec{\omega}$$

Clearly $\overleftrightarrow{I}$ is <u>symmetric</u>, $I_{jk} = I_{kj}$, with only <u>real</u> elements, i.e., $\overleftrightarrow{I}$ is <u>hermitean</u> (or self-adjoint, cf. QM). Then $\overleftrightarrow{I}$ can always be <u>diagonalized</u>, i.e., the body set of coords. can be transformed into a coord. system where $\overleftrightarrow{I}$ is diagonal:

$$\overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = \underline{\text{principal}} \text{ moment of inertia tensor}$$

x_1, x_2, x_3 = principal axes

$L_j = I_j \omega_j$ = angular momentum components $(j = 1,2,3)$ (No sum..!)

Ex: Symmetrical top (snurrebass): $I_1 = I_2 \neq I_3$

Spherical top: $I_1 = I_2 = I_3 = I \Rightarrow \vec{L} = I\vec{\omega}$

Kinetic energy for rotation around fixed point:

$$T = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2} m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i)$$

$$= \frac{1}{2} m_i \vec{\omega} \cdot (\vec{r}_i \times \vec{v}_i) \qquad \text{[Check yourself]}$$

$$= \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} \vec{\omega} \overleftrightarrow{I} \vec{\omega} = \frac{1}{2} \omega_j I_{jk} \omega_k$$

If rotation around axis $\hat{n}$, i.e., $\vec{\omega} = \omega \hat{n}$:

$$T = \frac{1}{2} \omega n_j I_{jk} \omega n_k = \frac{1}{2} (n_j I_{jk} n_k) \omega^2 = \frac{1}{2} I \omega^2$$

with $I \equiv n_j I_{jk} n_k = \hat{n} \overleftrightarrow{I} \hat{n}$ = mom. of inertia around rot. axis $\hat{n}$

In coord. system with principal axes x_1, x_2, x_3:

$$T = \frac{1}{2} I_j \omega_j^2$$

5.5 Euler equations of motion

From p.4 : $\vec{\tau} = \dot{\vec{L}}_s$ (measured in the inertial frame)

Measured in the rotating system (p.64; drop index b):

$$\underline{\vec{\tau} = \dot{\vec{L}} + \vec{\omega} \times \vec{L}}$$ (Euler eqns. with one point fixed)

Assume body axes along the principal axes such that $\overleftrightarrow{I}$ is diagonal and $L_j = I_j \omega_j$. Then:

$$\tau_i = \dot{L}_i + \varepsilon_{ijk} \omega_j L_k \quad ; \quad i = 1, 2, 3$$

Explicit:

$$\begin{aligned}
\tau_1 &= I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) \\
\tau_2 &= I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) \\
\tau_3 &= I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2)
\end{aligned}$$

5.6 Free rotation. Precession

Assume $\vec{\tau} = 0$ and symmetrical body with $I_1 = I_2 \neq I_3$.

$$\Rightarrow \begin{aligned}
I_1 \dot{\omega}_1 &= \omega_2 \omega_3 (I_1 - I_3) \\
I_2 \dot{\omega}_2 &= \omega_1 \omega_3 (I_3 - I_1) \\
I_3 \dot{\omega}_3 &= 0
\end{aligned}$$

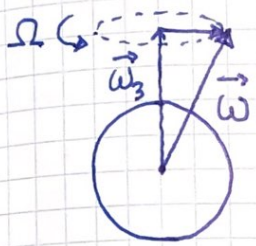
$\Rightarrow \omega_3 = \text{const.}$ (determined by initial conditions)

$$\Rightarrow \dot{\omega}_1 = -\Omega \omega_2 \,, \quad \dot{\omega}_2 = \Omega \omega_1 \quad ; \quad \Omega = \frac{I_3 - I_1}{I_1} \omega_3$$

$\Rightarrow \ddot{\omega}_1 + \Omega^2 \omega_1 = 0$ (after eliminating ω_2)

$$\Rightarrow \omega_1(t) = A \cos \Omega t \,, \quad \omega_2(t) = A \sin \Omega t$$

$\Rightarrow$ $\vec{\omega}$ precesses around the body x_3 axis, with angular frequency Ω and constant $\omega = |\vec{\omega}|$:



With $\vec{\tau} = 0$ both T and L^2 are conserved:

$$\left.\begin{aligned} T &= \tfrac{1}{2} I_1 A^2 + \tfrac{1}{2} I_3 \omega_3^2 \\ L^2 &= I_1^2 A^2 + I_3^2 \omega_3^2 \end{aligned}\right\} \Rightarrow A(T,L) \text{ and } \omega_3(T,L)$$

Rotating (spinning) planets: Chandler wobble)

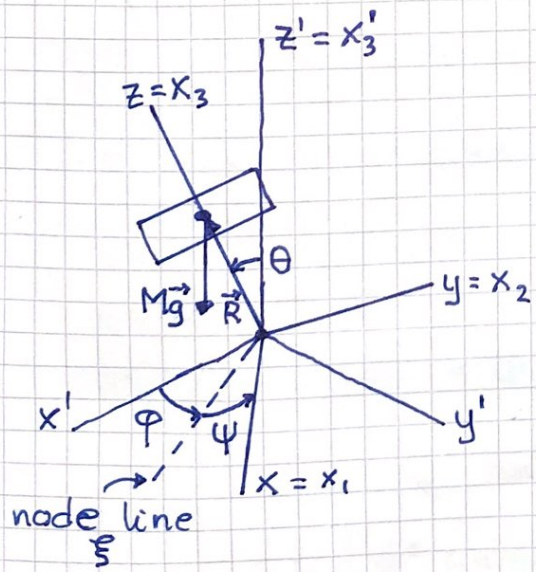
$I_3 - I_1 \ll I_1 \quad \Rightarrow \quad \Omega \ll \omega_3$

Earth: $2\pi/\omega_3 = 1$ day ; $2\pi/\Omega = 433$ days

Mars: $2\pi/\omega_3 \cong 1$ day 40 min ; $2\pi/\Omega = 207$ days

[A.S. Konopliv et al, Geophysical Research Letters, Oct. 2020]

5.7 Symmetrical top with one point fixed



The top spins around the x_3 axis (a principal axis) with the origin a fixed point.

External torque around origin:

$$\vec{\tau} = \vec{R} \times M\vec{g}$$

$$\left.\begin{aligned} x_j &\text{ fixed in the top} \\ x_j' &\text{ fixed in space} \end{aligned}\right\} \text{NB: } \underline{\text{Opposite}} \text{ in chapter 4!!}$$

$$\Rightarrow \vec{g} = -g\,\hat{x}_3' \;; \quad \vec{R} = \ell\,\hat{z} = \ell\,\hat{x}_3 \quad (\ell = |\vec{R}|)$$

Symmetry $\Rightarrow$ $I_1 = I_2 \neq I_3$

Kinetic energy: $T = \frac{1}{2} I_j \omega_j^2 = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$

1 point fixed $\Rightarrow$ 3 independent degrees of freedom

Will use the Euler angles φ, θ, ψ to construct the Lagrangian $L = T - V$ $\Rightarrow$ Need $\omega_j(q_i, \dot{q}_i)$, i.e., expressions for the components of $\vec{\omega}$ along the body axes x_j $(j = 1,2,3)$. We transform from space to body by considering $\vec{\omega}''$ as 3 successive rotations with angular velocities $\omega''_\varphi = \dot{\varphi}$, $\omega'_\theta = \dot{\theta}$ and $\omega'_\psi = \dot{\psi}$.

$$\vec{\omega}''_\varphi = \dot{\varphi}\,\hat{x}''_3 \rightarrow \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} \Rightarrow \vec{\omega}_\varphi = \mathbb{A}\,\vec{\omega}''_\varphi = \mathbb{B}\mathbb{C}\mathbb{D}\,\vec{\omega}''_\varphi$$

With $\mathbb{A}$ from p. 61:

$(\omega_\varphi)_1 = \dot{\varphi}\sin\theta\sin\psi$, $(\omega_\varphi)_2 = \dot{\varphi}\sin\theta\cos\psi$, $(\omega_\varphi)_3 = \dot{\varphi}\cos\theta$

$$\vec{\omega}'_\theta = \dot{\theta}\,\hat{\xi} \rightarrow \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{\omega}_\theta = \mathbb{B}\,\vec{\omega}'_\theta$$

With $\mathbb{B}$ from p. 60:

$(\omega_\theta)_1 = \dot{\theta}\cos\psi$, $(\omega_\theta)_2 = -\dot{\theta}\sin\psi$, $(\omega_\theta)_3 = 0$

$$\vec{\omega}''_\psi = \dot{\psi}\,\hat{x}_3 \rightarrow \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \Rightarrow \vec{\omega}_\psi = \vec{\omega}''_\psi$$

$(\omega_\psi)_1 = 0$, $(\omega_\psi)_2 = 0$, $(\omega_\psi)_3 = \dot{\psi}$

We add contributions to each component ω_j:

$$\omega_1 = (\omega_\varphi)_1 + (\omega_\theta)_1 + (\omega_\psi)_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}$$

$$\Rightarrow \omega_1^2 + \omega_2^2 = \dot{\varphi}^2\sin^2\theta + \dot{\theta}^2$$

$$\Rightarrow T = \tfrac{1}{2} I_1 (\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2} I_3 (\dot{\varphi}\cos\theta + \dot{\psi})^2$$

Pot. energy: $V = Mg\ell\cos\theta$ [$V = 0$ in $x'y'$ plane]

Lagrangian: $L = T - V = L(\theta, \dot\varphi, \dot\theta, \dot\psi)$

$\Rightarrow \varphi, \psi$ cyclic coords. $\Rightarrow p_\varphi, p_\psi$ conserved:

$$\left.\begin{aligned} p_\psi &= \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\varphi\cos\theta) = I_3\omega_3 \equiv I_1 a \\ p_\varphi &= \frac{\partial L}{\partial \dot\varphi} = (I_1\sin^2\theta + I_3\cos^2\theta)\dot\varphi + I_3\dot\psi\cos\theta \equiv I_1 b \end{aligned}\right\} \text{These define } a \text{ and } b$$

Conservative system $\Rightarrow$ energy is conserved:

$$E = T + V = \tfrac{1}{2}I_1(\dot\theta^2 + \dot\varphi^2\sin^2\theta) + \underbrace{\tfrac{1}{2}I_3\omega_3^2}_{\text{const.}} + Mgl\cos\theta$$

From p_ψ: $I_3\dot\psi = I_1 a - I_3\dot\varphi\cos\theta$

Insert this in expression for p_φ:

$I_1\dot\varphi\sin^2\theta + I_1 a\cos\theta = I_1 b$ [I_3 terms cancel]

$$\Rightarrow \dot\varphi = \frac{b - a\cos\theta}{\sin^2\theta} \quad ; \quad \dot\psi = \frac{I_1 a}{I_3} - \frac{b - a\cos\theta}{\sin^2\theta}\cdot\cos\theta$$

$$\Rightarrow E' \equiv E - \frac{I_3}{2}\omega_3^2 = \frac{I_1}{2}\dot\theta^2 + \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

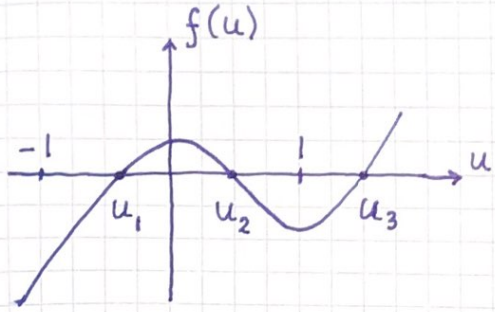
A one-dimensional problem in the variable θ with an effective potential

$$V'(\theta) = Mgl\cos\theta + \tfrac{1}{2}I_1\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2$$

The substitutions $u = \cos\theta$, $\alpha = 2E'/I_1$ and $\beta = 2Mgl/I_1$ and some algebra yields the equation of motion

$$\dot u^2 = \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2) \equiv f(u) \geq 0$$

Solution of $\dot u^2 = f(u)$ yields $\theta(t)$, the <u>nutation</u> of the x_3 axis relative to the vertical space axis x_3'. Next, $\theta(t)$ determines $\dot\varphi(t)$, the <u>precession</u> of x_3 around x_3', and $\dot\psi(t)$, the <u>spinning</u> around x_3.



Solutions of $f(u)=0$ corresponds to $\dot{u}=0$ or $\dot{\theta} = -\dot{u}/\sin\theta = 0$, i.e., values of θ where $\dot{\theta}$ changes sign.

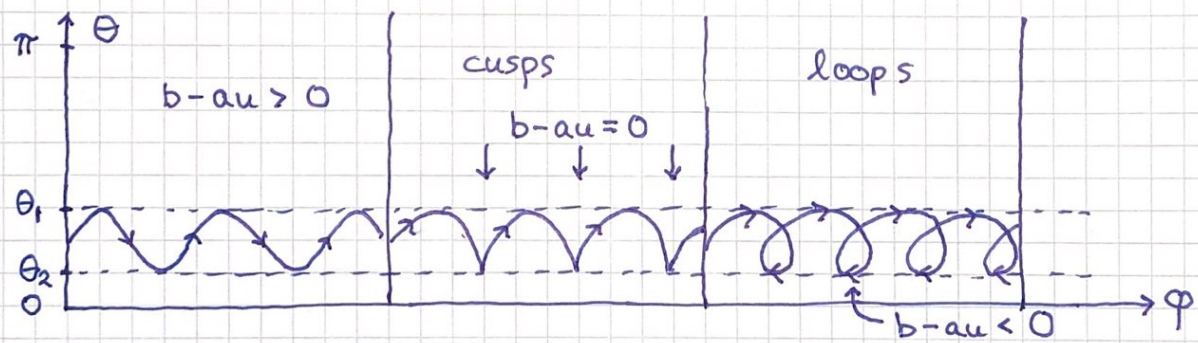
$$\left.\begin{aligned} f(\pm 1) &= -(b \mp a)^2 < 0 \\ f(\pm\infty) &= \pm\infty \end{aligned}\right\}$$ always a solution $u_3 > 1$ which is not physical

Acceptable values of $u = \cos\theta$: $\quad u_1 < u < u_2$

Precession: $\dot{\varphi} = \dfrac{b - a\cos\theta}{\sin^2\theta} = \dfrac{b - au}{1-u^2}$

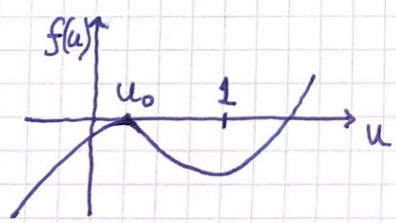
Usually $\dot{\psi} \gg \dot{\varphi}$ for a spinning top.

May have nutation, $\theta_2 < \theta < \theta_1$, with $\dot{\varphi} > 0$ always, $\dot{\varphi} \gtrsim 0$ (cusps) or $\dot{\varphi} \lesssim 0$ (loops):



Regular precession: $\theta(t) = \theta_0 = \text{const.}$

$\Rightarrow f(u) = 0$ for $u = u_1 = u_2 = u_0 = \cos\theta_0$

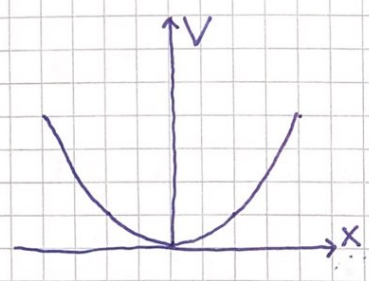


$f = 0$ and $df/du = 0$ in u_0 (starting point for a deeper analysis of regular prec.)

6. Oscillations

Applications:
Vibrations in molecules, solids and mechanical systems
Oscillations in coupled electrical circuits
Acoustics

Brief review of 1D harmonic oscillator:



$$V(x) = \frac{1}{2}kx^2$$

Curvature $k = \partial^2 V/\partial x^2$

N2: $F = m\ddot{x}$ with $F = -\partial V/\partial x = -kx$

$$\Rightarrow m\ddot{x} + kx = 0 \quad \Rightarrow x(t) = \mathrm{Re}\{A e^{-i\omega_0 t}\} \; ; \quad \omega_0 = \sqrt{k/m}$$

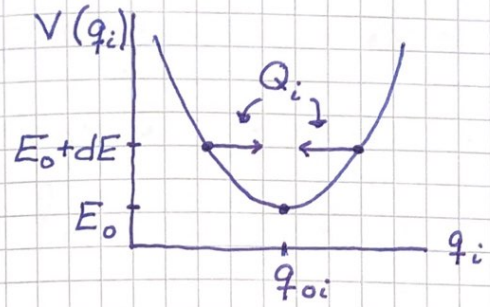
We will generalize the simple harm. osc. to a system of coupled harm. osc. Assume a conservative system with $n = 3N - k$ degrees of freedom (i.e. N particles and k holonomic constraints), where we have transformed from $\vec{r}_j$ $(j = 1, .., N)$ to generalized independent coords. q_i $(i = 1, .., n)$.

Equilibrium when all generalized forces are zero,

$$Q_i = -(\partial V/\partial q_i)_0 = 0$$

Equilibrium configuration: q_{0i} $(i = 1, .., n)$

Stable equilibrium:



Small disturbance from equilibrium $\Rightarrow$ Small oscillation around equilibrium

$$q_i(t) = q_{0i} + \eta_i(t)$$

η_i = deviation from eq. = new generalized coords.

Taylor expansion of V around eq.: [Sum conv.!]

$$V(\{q_i\}) = V(\{q_{0i}\}) + \left(\frac{\partial V}{\partial q_i}\right)_0 \eta_i + \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j + \dots$$

Eq. condition: $(\partial V/\partial q_i)_0 = 0$

Free choice of $V=0$, e.g., $V(\{q_{0i}\}) = 0$

$$\Rightarrow \quad V = \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j \equiv \frac{1}{2} V_{ij} \eta_i \eta_j$$

with symmetric matrix $\mathbb{V}$, i.e., $V_{ij} = V_{ji}$

Kinetic energy T is quadratic in velocities,

$$T = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} m_{ij} \dot{\eta}_i \dot{\eta}_j$$

with $m_{ij}(\{q_i\}) = m_{ij}(\{q_{0i}\}) + (\partial m_{ij}/\partial q_k)_0 \eta_k + \dots$

$$\approx m_{ij}(\{q_{0i}\})$$

since T is already quadratic in (the small) $\dot{\eta}_i$

$$\Rightarrow T = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j \quad ; \quad T_{ij} \equiv m_{ij}(\{q_{0i}\})$$

with symmetric $\mathbb{T}$, i.e., $T_{ij} = T_{ji}$ [Usually $T_{ij} = T_i \delta_{ij}$]

Lagrangian : $L = T - V = \frac{1}{2}\left(T_{ij}\,\dot{\eta}_i\dot{\eta}_j - V_{ij}\,\eta_i\eta_j\right)$

Eqns. of motion : $\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_i} - \frac{\partial L}{\partial \eta_i} = 0 \quad ; \quad i = 1,..,n$

Consider <u>particular</u> i, with sum indices ℓ and j :

$$\frac{\partial L}{\partial \dot{\eta}_i} = \frac{1}{2}T_{\ell j}\frac{\partial}{\partial \dot{\eta}_i}\left(\dot{\eta}_\ell\dot{\eta}_j\right) = \frac{1}{2}T_{\ell j}\left(\delta_{i\ell}\dot{\eta}_j + \delta_{ij}\dot{\eta}_\ell\right)$$

$$= \frac{1}{2}T_{ij}\dot{\eta}_j + \frac{1}{2}T_{\ell i}\dot{\eta}_\ell = T_{ij}\dot{\eta}_j$$

(using $T_{\ell i} = T_{i\ell}$ and renaming sum index $\ell \to j$)

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_i} = T_{ij}\frac{d}{dt}\dot{\eta}_j = T_{ij}\ddot{\eta}_j$$

Similarly :

$$\frac{\partial L}{\partial \eta_i} = -\frac{\partial V}{\partial \eta_i} = \ldots = -V_{ij}\,\eta_j$$

$$\Rightarrow \quad T_{ij}\ddot{\eta}_j + V_{ij}\,\eta_j = 0 \quad ; \quad i = 1,\ldots,n$$

A set of n coupled linear 2. order diff. eqns.
Same <u>form</u> as 1D harm. osc.

$\Rightarrow$ Reasonable ansatz is : $\eta_i(t) = A_i\, e^{-i\omega t}$
with complex amplitudes A_i, and $\mathrm{Re}\,\eta_i$ = actual movement

$$\Rightarrow V_{ij}A_j - \omega^2 T_{ij}A_j = 0$$

Non-trivial solution when $\det\left(\mathbb{V} - \omega^2\mathbb{T}\right) = 0$,
an nth order eqn. for ω^2, with solutions (roots)
ω_α^2 ; $\alpha = 1,..,n$, the <u>eigenfrequencies</u> of the system.

Eigenmodes α :

$A_{i\alpha}$ = amplitude along gen. coord. q_i in mode α

Determined by the set of eqns.

$$(V_{ij} - \omega_\alpha^2 T_{ij}) A_{j\alpha} = 0$$

For a given ω_α, these n eqns. determine $n-1$ components of

$$\vec{A}_\alpha = \begin{pmatrix} A_{1\alpha} \\ A_{2\alpha} \\ \vdots \\ A_{n\alpha} \end{pmatrix}$$

[Cf. Assignment 2, Question 2, double pendulum ; $n=2$]

$\Rightarrow$ <u>One</u> undetermined <u>complex</u> component for each mode α, e.g. $A_{1\alpha}$, with undetermined abs. value and phase

$\Rightarrow$ $2n$ undetermined factors, as expected for n second order diff. eqns.

Gen. coords. for oscillation in eigenmode α :

$$\eta_{j\alpha} = A_{j\alpha} e^{-i\omega_\alpha t}$$

General solution is a linear combination of eigenmodes :

$$\eta_j(t) = \sum_{\alpha=1}^{n} \eta_{j\alpha}(t) = \sum_{\alpha=1}^{n} A_{j\alpha} e^{-i\omega_\alpha t}$$

Actual movement :

$$\mathrm{Re}\, \eta_j(t) = \sum_{\alpha=1}^{n} \mathrm{Re}\{ A_{j\alpha} e^{-i\omega_\alpha t} \}$$

Normal modes :

$A_{j\alpha}$ is prop. with the cofactor $\Delta_{j\alpha} = (-1)^{j+\alpha} \cdot M_{j\alpha}$ of the determinant $|\mathbb{V} - \omega_\alpha^2 \mathbb{T}|$. The minor (subdeterminant) $M_{j\alpha}$ is evaluated by removing row j and column α in $|\mathbb{V} - \omega_\alpha^2 \mathbb{T}|$.

$$\Rightarrow \operatorname{Re} \eta_j(t) = \sum_\alpha \Delta_{j\alpha} \operatorname{Re}\{c_\alpha e^{-i\omega_\alpha t}\} \equiv \sum_\alpha \Delta_{j\alpha} \Theta_\alpha(t)$$

In these <u>normal</u> coordinates Θ_α (and with a suitable scaling of $\Delta_{j\alpha}$) we have

$$T = \frac{1}{2} \sum_\alpha \dot{\Theta}_\alpha^2 \quad ; \quad V = \frac{1}{2} \sum_\alpha \omega_\alpha^2 \Theta_\alpha^2$$

$$\Rightarrow L = \frac{1}{2} \sum_\alpha \left(\dot{\Theta}_\alpha^2 - \omega_\alpha^2 \Theta_\alpha^2 \right)$$

Finally, with $\frac{d}{dt} \frac{\partial L}{\partial \dot{\Theta}_\alpha} - \frac{\partial L}{\partial \Theta_\alpha} = 0$:

$$\ddot{\Theta}_\alpha + \omega_\alpha^2 \Theta_\alpha = 0 \quad ; \quad \alpha = 1, \dots, n$$

Comments :

- Decoupled equations, i.e., in some sense orthogonal, hence the name normal modes
- n 2nd order eqns. require $2n$ initial conditions for a complete solution, i.e., to determine abs. value and phase of the complex constants $c_\alpha = |c_\alpha| \exp(i\varphi_\alpha)$
- See compendium for more details (proof)

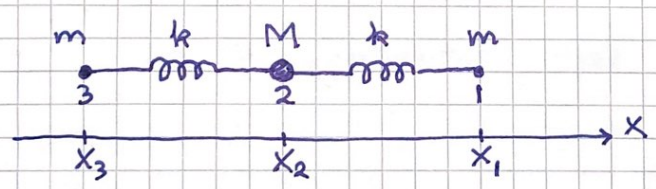
Ex. 1: Particle in 3D harmonic potential

$$V = \frac{1}{2}\left(k_1 x^2 + k_2 y^2 + k_3 z^2\right)$$
$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$
$$L = T - V = \frac{1}{2}\sum_{i=1}^{3}\left(m\dot{x}_i^2 - k_i x_i^2\right)$$
$$\Rightarrow m\ddot{x}_i + k_i x_i = 0 \quad ; \quad i = 1, 2, 3 \qquad \text{(No sum here)}$$
$$\Rightarrow x_i = A_i e^{-i\omega_i t} \quad \text{with} \quad \omega_i = \sqrt{k_i/m} \quad ; \quad i = 1, 2, 3$$

Here, $\mathbb{V}$ and $\mathbb{T}$ are diagonal from the start, so the cartesian coords. x_i are the normal coords.

<u>Degeneracy</u> if the force field is central, $V = V(r)$, i.e., $k_1 = k_2 = k_3 = k$ and $V(r) = \frac{1}{2}kr^2$. Then $\omega_1 = \omega_2 = \omega_3 = \sqrt{k/m}$

Ex. 2: Linear symmetric 3-atomic molecule



(Ex: CO_2)

Equilibrium bond lengths: $x_{01} - x_{02} = x_{02} - x_{03} = b$

Deviations from equilibrium: $\eta_i = x_i - x_{0i} \quad (i = 1, 2, 3)$

Assume nearest neighbour interaction and harmonic potential:

$$V = \frac{1}{2}k\left[(x_1 - x_2) - b\right]^2 + \frac{1}{2}k\left[(x_2 - x_3) - b\right]^2$$
$$= \frac{1}{2}k(\eta_1 - \eta_2)^2 + \frac{1}{2}k(\eta_2 - \eta_3)^2 = \frac{1}{2}V_{ij}\eta_i\eta_j$$
$$T = \frac{1}{2}m_i\dot{x}_i^2 = \frac{1}{2}m\left(\dot{\eta}_1^2 + \dot{\eta}_3^2\right) + \frac{1}{2}M\dot{\eta}_2^2 = \frac{1}{2}T_{ij}\dot{\eta}_i\dot{\eta}_j$$
$$\mathbb{V} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \quad ; \quad \mathbb{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

Eigenfreqs. given by $\left|\mathbb{V} - \omega^2\mathbb{T}\right| = 0$

$$\Rightarrow \begin{vmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 m \end{vmatrix} = 0$$

$$\Rightarrow \ldots \Rightarrow \quad \omega^2 \left(k-\omega^2 m\right)\left(-k(2m+M) + \omega^2 m M\right) = 0$$

$$\Rightarrow \omega_1 = 0, \quad \omega_2 = \sqrt{\frac{k}{m}}, \quad \omega_3 = \sqrt{\frac{(2m+M)k}{mM}}$$

The analysis of the atomic movements in the various normal modes $\alpha = 1, 2, 3$ is left as an exercise.

- $\omega_1 = 0$ corresponds to translation of the molecule along the x axis, with

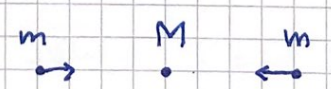
$$\vec{A}_1 \approx \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- ω_3 is the antisymmetric mode with

$$\vec{A}_3 \sim \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$$

- ω_2 is the symmetric mode with

$$\vec{A}_2 \sim \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



The "recipe" on p. 77 works for ω_1 and ω_3. For ω_2, $M_{12} = M_{22} = M_{32} = 0$. Then, one must simply solve the set of eqns.

$$\left(V_{ij} - \omega_\alpha^2 T_{ij}\right) A_{j\alpha} = 0$$

Here, 3 eqns. for A_{12}, A_{22} and A_{32}.

7. Special theory of relativity

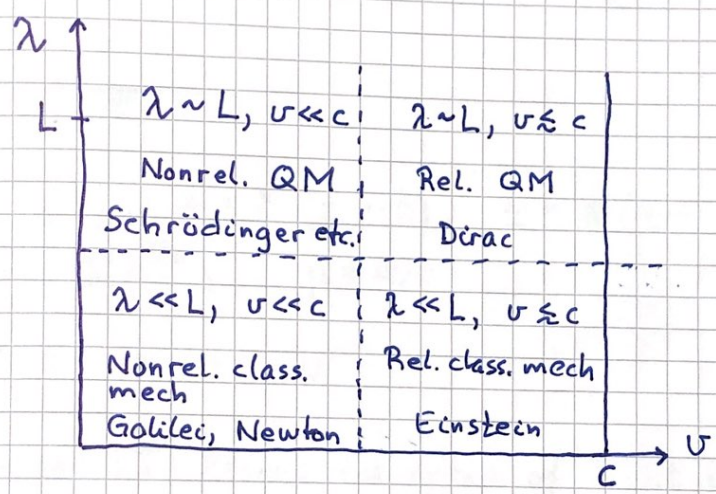
7.1 Introduction

L = linear size of system

$\lambda = h/p$ = de Broglie wavelength for particle (body)
(p = momentum, h = Planck constant)

v = typical particle velocity

c = speed of light



We will consider systems with $\lambda \ll L$ and $v \lesssim c$, described by Einstein's special theory of relativity (SR)
[See GPS 7.11 for a very brief outline of the general theory (GR)]

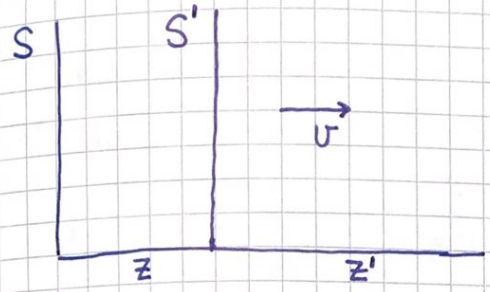
Galilei: Assume two inertial systems S and S'; S' moves with velocity $\vec{v}$ relative to S. A particle's coordinates $\vec{r}$ and $\vec{r}'$ and times t and t' observed in S and S' are related as $\vec{r}' = \vec{r} - \vec{v} \cdot t$ and $t' = t$. (GT)
The speed of light is different in S and S', $c' = c - v$.

Experiments: $c' = c$ [Michelson - Morley etc]

Einstein's two postulates:

1. The laws of physics are the same in all inertial systems
2. The speed of light is ——— " ———

7.2 The Lorentz transformation (LT)



Assume common origin at $t = t' = 0$ and $\vec{v} = v\hat{z}$

LT: $x' = x$, $y' = y$, $z' = \gamma(z - vt)$, $t' = \gamma\left(t - \frac{v}{c^2} z\right)$

$\gamma = (1 - \beta^2)^{-1/2}$, $\beta = v/c$

Inverse LT: $v \to -v$

Will show that LT leaves the speed of light invariant.
Consider source of light wave in origo of S' at $t' = 0$.
Equation for the light front:

$$r' = ct' \Rightarrow (r')^2 = (ct')^2 \Rightarrow (x')^2 + (y')^2 + (z')^2 = c^2 (t')^2$$

Use LT to find eqn. for light front in S:

$$(x')^2 + (y')^2 + (z')^2 - c^2 (t')^2$$
$$= x^2 + y^2 + \gamma^2 (z - vt)^2 - c^2 \gamma^2 \left(t - \frac{v}{c^2} z\right)^2$$
$$= \ldots \text{ some algebra} \ldots = x^2 + y^2 + z^2 - c^2 t^2$$

$\Rightarrow$ Light front has speed c in both S' and S

If $v \ll c$: $x' = x$, $y' = y$, $z' = z - vt$, $t' = t$; LT $\xrightarrow{v \ll c}$ GT

Inverse transf.: $x = x'$, $y = y'$, $z = \gamma(z' + vt')$, $t = \gamma\left(t' + \frac{v}{c^2} z'\right)$

[Obvious: S moves with speed $-v$ relative to S']

LT with vectors $(\vec{v} = c\vec{\beta})$:

$$\vec{r}' = \vec{r} + (\gamma - 1)\frac{(\vec{\beta}\cdot\vec{r})\,\vec{\beta}}{\beta^2} - \gamma c t \vec{\beta}$$

$$t' = \gamma t - \frac{\gamma}{c}\vec{\beta}\cdot\vec{r}$$

LT is linear $\Rightarrow$ may use ideas/results from chapter 4

Minkowski space:

4D space with axes $x_1 = x, x_2 = y, x_3 = z$ and $x_4 = ict$

$$\Rightarrow x^2 + y^2 + z^2 - c^2t^2 = \sum_{\mu=1}^{4} x_\mu x_\mu \equiv x_\mu x_\mu \quad \text{[Sum conv.]}$$

Index notation: Greek for 4-vectors $(\alpha, \beta, \mu \ldots)$
Roman for 3-vectors $(i, j, k \ldots)$

$\Rightarrow$ Invariance of the speed of light can be expressed as

"$x_\mu x_\mu$ is an invariant"

Or: The norm (abs. value) of the "position vector" in Minkowski space is constant under a LT

From 4.2 (p. 55): A linear transf. that leaves the norm of the vector unchanged is an orthogonal transf.

$\Rightarrow$ LT is an orthogonal transf. in Minkowski space

LT on matrix form:

$$\mathbb{x}' = \mathbb{L}\,\mathbb{x} \quad \text{or} \quad x'_\mu = L_{\mu\nu} x_\nu$$

With relative velocity along x_3 axis $(\vec{v} = v\hat{x}_3)$:

$$\mathbb{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}$$

From 4.3 (p. 58) : $\mathbb{L}^{-1} = \tilde{\mathbb{L}}$ (= transposed of $\mathbb{L}$)

Inverse LT : $\mathbb{X} = \mathbb{L}^{-1}\mathbb{X}' = \tilde{\mathbb{L}}\mathbb{X}' \Rightarrow x_\mu = L_{\nu\mu} x'_\nu$

$$\mathbb{L}^{-1} = \tilde{\mathbb{L}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{pmatrix}$$

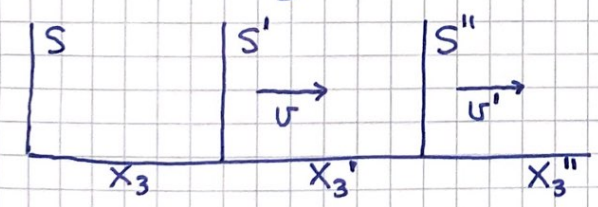
[$v \to -v$ as seen before]

Trigonometric convenience :

From example p. 56 : $\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ repr. rotation in a plane

Then, with $\gamma = \cos\varphi$ and $i\beta\gamma = \sin\varphi$, a LT can be viewed as a rotation an <u>imaginary</u> angle φ in the $x_3 x_4$ plane.

Ex : Velocity addition



v = speed of S' rel. to S
v' = —"— S'' —"— S'
v'' = —"— S'' —"— S
$v'' = ?$ GT : $v'' = v' + v$

$$\mathbb{X}'' = \mathbb{L}''\mathbb{X}' = \mathbb{L}''\mathbb{L}\mathbb{X} = \mathbb{L}''\mathbb{X}$$

$$\Rightarrow L''_{33} = \cos\varphi'\cos\varphi - \sin\varphi'\sin\varphi = \cos(\varphi'+\varphi) = \cos\varphi''$$

$$L''_{34} = \cos\varphi'\sin\varphi + \sin\varphi'\cos\varphi = \sin(\varphi'+\varphi) = \sin\varphi''$$

$$\Rightarrow \tan\varphi'' = \frac{L''_{34}}{L''_{33}} = \dots = \frac{\tan\varphi + \tan\varphi'}{1 - \tan\varphi\tan\varphi'}$$

$$\Rightarrow i\beta'' = \frac{i\beta + i\beta'}{1 + \beta\beta'} \Rightarrow \underline{v'' = \frac{v + v'}{1 + vv'/c^2}}$$; Einstein's addition formula

7.3 Covariant four-dimensional formulation

We will address Einstein's 1. postulate (the principle of relativity):

Are the laws of physics the same in all inertial systems?

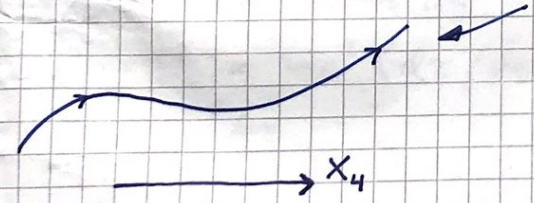
Or:

Are the equations (describing the laws of physics) covariant, i.e., invariant in form, under a LT?

Laws-of-physics-equations provide the relation between scalars, vectors and tensors (expressed by matrices). Scalars and vectors are special cases of tensors. Hence, we must in general have <u>covariance of tensor equations</u>:

If $C_{\mu\nu} = D_{\mu\nu}$, then $C'_{\mu\nu} = D'_{\mu\nu}$ after a LT

Consider a particle (or observer), moving in 3D space as time goes by; $u^2 = (dx_i/dt)\cdot(dx_i/dt)$:



The particle's world line (or: proper line) in Minkowski space (or: spacetime)

dx_μ = small element along the world line

Proper time τ : $dx_\mu\, dx_\mu \equiv -c^2\, d\tau^2$ (def. of τ)

Let us show that

$$dx'_\mu\, dx'_\mu = dx_\mu\, dx_\mu$$

i.e., the same in S' and S, i.e., a Lorentz invariant

From 7.2 (and 4.3):

$$\mathbb{X}' = \mathbb{L}\,\mathbb{X} \; ; \quad x'_\mu = L_{\mu\nu} x_\nu$$

$$\mathbb{L}^{-1} = \tilde{\mathbb{L}} \quad \Rightarrow \mathbb{L}\tilde{\mathbb{L}} = \tilde{\mathbb{L}}\mathbb{L} = \mathbb{1} \quad (\text{since } \mathbb{L}\mathbb{L}^{-1} = \mathbb{L}^{-1}\mathbb{L} = \mathbb{1})$$

Then:

$$\begin{aligned} dx'_\mu dx'_\mu &= L_{\mu\nu}\, dx_\nu \cdot L_{\mu\alpha}\, dx_\alpha \\ &= L_{\mu\nu} \tilde{L}_{\alpha\mu}\, dx_\nu\, dx_\alpha \\ &= \delta_{\alpha\nu}\, dx_\nu\, dx_\alpha = dx_\alpha\, dx_\alpha = dx_\mu\, dx_\mu \end{aligned}$$

I.e., $dx'_\mu dx'_\mu = dx_\mu dx_\mu = -c^2 d\tau^2$

Time dilation:

Assume particle at rest in S'; $u = v =$ speed of S' rel. to S

$\Rightarrow dx'_\mu = (0, 0, 0, ic\,dt')$

$\Rightarrow dx'_\mu dx'_\mu = -c^2 dt'^2 = -c^2 d\tau^2$

$\Rightarrow d\tau = dt' =$ time measured on clock at rest

Measured in S: $dx_\mu = (dx, dy, dz, ic\,dt)$

$\Rightarrow dx_\mu dx_\mu = dx^2 + dy^2 + dz^2 - c^2 dt^2 = -c^2 d\tau^2$

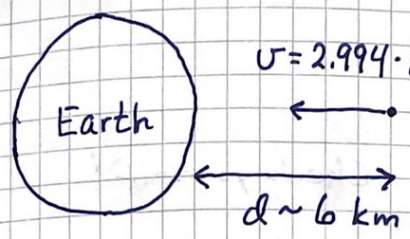
$\Rightarrow u^2 - c^2 = -c^2 (d\tau/dt)^2$

$$\Rightarrow dt = \frac{d\tau}{\sqrt{1-\beta^2}} > d\tau \qquad (\beta = u/c = v/c)$$

$\Rightarrow dt = \gamma\, d\tau =$ time measured on a clock in S, where the particle moves with speed v, is longer than the time measured on a clock in S', where the particle is at rest

Or: Moving clocks run slow.

Ex: Lifetime of muons in the atmosphere



Nonrel.: $l = v \cdot \tau \approx 600\,m \ll d$

Rel.: $t = \gamma \tau \approx 32\ \mu s$ = lifetime measured on earth

$l = v \cdot t \approx 9.6\ km > d$

$\Rightarrow$ Muons created in the atmosphere arrive on earth during their lifetime, in agreement with experiment.
[Bailey et al, Physics Letters 55B, 420 (1975)]

Classification of 4-vectors:

space-like : $x_\mu x_\mu > 0$
time-like : $x_\mu x_\mu < 0$
light-like : $x_\mu x_\mu = 0$

Assume S' has speed v along $\hat{z}$ relative to S. Consider events 1 and 2 described by $x_{1\mu}$ and $x_{2\mu}$ in S. The difference $X_\mu = x_{1\mu} - x_{2\mu}$ has norm

$$X_\mu X_\mu = (z_1 - z_2)^2 - c^2 (t_1 - t_2)^2$$

The time difference in S' is

$$t_1' - t_2' = \gamma \left[t_1 - t_2 - \frac{v}{c^2}(z_1 - z_2) \right] = \frac{\gamma}{c} \left[c(t_1 - t_2) - \frac{v}{c}(z_1 - z_2) \right]$$

space-like $X_\mu \Rightarrow X_\mu X_\mu > 0 \Rightarrow c < \left|\frac{z_1 - z_2}{t_1 - t_2}\right|$

$\Rightarrow$ 1 and 2 cannot be connected with a signal with speed c

$\Rightarrow$ the two events cannot influence each other

However, we can find a $v \leq c$ that gives

$$t_1' - t_2' = 0 \quad \text{or} \quad t_1' - t_2' \gtrless 0$$

I.e., before and after are not uniquely determined.

time-like $X_\mu \Rightarrow X_\mu X_\mu < 0 \Rightarrow c > \left|\frac{z_1 - z_2}{t_1 - t_2}\right|$

$\Rightarrow$ 1 and 2 can be connected with a signal with speed c

$\Rightarrow$ the two events can influence each other

Now, $t_1' - t_2'$ and $t_1 - t_2$ have the same sign

I.e., before and after are uniquely determined

Causality : Cause before effect.

4-velocity : $u_\mu \equiv dx_\mu / d\tau = (dx_i, ic\,dt)/d\tau$

$$\Rightarrow u_i = dx_i/d\tau = \gamma\, dx_i/dt = \gamma v_i$$

$$u_4 = dx_4/d\tau = ic\, dt/d\tau = ic\gamma$$

$$\Rightarrow u_\mu = \gamma(\vec{v}, ic)$$

Time-like :

$$u_\mu u_\mu = \gamma^2 v^2 - \gamma^2 c^2 = \frac{v^2 - c^2}{1 - v^2/c^2} = -c^2 < 0$$

4-current density :

The continuity equation is $\nabla \cdot \vec{j} + \partial\rho/\partial t = 0$

with $\vec{j}$ = current density, ρ = charge density

With $j_\mu = (\vec{j}, ic\rho)$: $\frac{\partial}{\partial x_\mu} j_\mu = 0$ or $\partial_\mu j_\mu = 0$

Due to length contraction, the charge density is larger for moving charges than for charges at rest:

$$\rho = \gamma \rho_0 > \rho_0$$

Hence,

$$j_\mu = (\vec{j}, ic\rho) = (\rho\vec{v}, ic\rho) = (\gamma\rho_0\vec{v}, ic\gamma\rho_0) = \rho_0 u_\mu$$

Let us show that the continuity eqn. is covariant. We must show that $\partial/\partial x_\mu$ transforms like dx_μ under a LT. We have (from p. 83) $dx_\mu = \underbrace{L_{\nu\mu}}_{= \partial x_\mu/\partial x'_\nu} dx'_\nu$

$$\Rightarrow \frac{\partial}{\partial x'_\nu} = \frac{\partial x_\mu}{\partial x'_\nu}\frac{\partial}{\partial x_\mu} = L_{\nu\mu}\frac{\partial}{\partial x_\mu}$$

$\Rightarrow$ Both $\partial/\partial x_\mu$ and $j_\mu = \rho_0 u_\mu$ transform as dx_μ under a LT

$\Rightarrow$ $\partial_\mu j_\mu$ is a Lorentz invariant scalar

$\Rightarrow$ The cont. eqn. <u>is</u> covariant

Maxwell's equations and the 4-potential:

From 1.5; assuming vacuum ($\vec{P} = \vec{M} = 0$):

$$\vec{B} = \nabla \times \vec{A} \quad ; \quad \vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$$

(using $\nabla\cdot\vec{B} = 0$ and $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$)

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \Rightarrow \nabla^2\phi + \frac{\partial}{\partial t}\nabla\cdot\vec{A} = -\rho/\varepsilon_0$$

$$\nabla\times\vec{B} = \mu_0\vec{j} + \varepsilon_0\mu_0\,\partial\vec{E}/\partial t \quad ; \quad \varepsilon_0\mu_0 = 1/c^2$$

$$\Rightarrow \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla\left(\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\mu_0\vec{j}$$

(using the identity $\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$; see Assignment 2, Question 3)

Maxwell's 4 eqns. for $\vec{E}$ and $\vec{B}$ are reduced to 2 coupled eqns. for ϕ and $\vec{A}$; these may be decoupled since we have some <u>freedom</u> in our choice for ϕ and $\vec{A}$:

$\vec{B} = \nabla \times \vec{A} \quad \Rightarrow \quad \vec{A} \to \vec{A} + \nabla \chi$ leaves $\vec{B}$ unchanged for arbitrary scalar χ, since $\nabla \times \nabla \chi = 0$

Since $\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t$, we must also let $\phi \to \phi - \partial\chi/\partial t$, to leave $\vec{E}$ unchanged.

This freedom to adjust ϕ and $\vec{A}$ is called <u>gauge invariance</u>.

Lorenz gauge (Ludvig Lorenz, Danish physicist, 1829-1891):

Choose a transformation that obeys

$$\nabla^2 \chi - \frac{1}{c^2}\partial^2\chi/\partial t^2 = 0$$

Then, the Lorenz condition

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\partial\phi/\partial t = 0$$

is gauge invariant,

$$\nabla \cdot (\vec{A} + \nabla\chi) + \frac{1}{c^2}\frac{\partial}{\partial t}(\phi - \partial\chi/\partial t)$$

$$= \underbrace{\nabla \cdot \vec{A} + \frac{1}{c^2}\partial\phi/\partial t}_{=0} + \underbrace{\nabla^2\chi - \frac{1}{c^2}\partial^2\chi/\partial t^2}_{=0} = 0$$

The Lorenz condition decouples the eqns. for $\vec{A}$ and ϕ:

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = -\mu_0\vec{j}$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\rho/\varepsilon_0$$

The 4-potential: $A_\mu = (\vec{A}, i\phi/c)$

$\Rightarrow$ Lorenz condition: $\partial_\mu A_\mu = 0$ [since $\frac{\partial}{\partial x_4} = \frac{\partial}{ic\partial t}$]

Eqn. for A_μ: $\Box^2 A_\mu = -\mu_0 j_\mu$; $\Box^2 \equiv \partial_\mu\partial_\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ = D'Alembert's operator

$\Rightarrow$ A_μ transforms like $j_\mu = (\vec{j}, ic\rho)$ and u_μ and x_μ under a Lorentz transf. (Hendrik Lorentz, Dutch physicist, 1853-1928)

$\Rightarrow$ Maxwell's eqns. are covariant, i.e., consistent with SR!

7.4 Relativistic dynamics

N2, $F = ma$, is invariant under a GT, not under a LT

$\Rightarrow$ We must generalize Newtonian dynamics such that it

(a) becomes covariant, and

(b) reduces to $\frac{d}{dt}(mv_i) = F_i$ when $v_i \ll c$

Natural attempt:

$$\frac{d}{d\tau}(m u_\mu) = K_\mu \quad \text{with} \quad \begin{cases} m = \text{invariant mass} \\ \tau = \text{proper time} \\ u_\mu = \text{4-velocity} \\ K_\mu = \text{Minkowski force} \end{cases}$$

Since $d\tau = dt/\gamma \to dt$ and $u_i = \gamma v_i \to v_i$ when $v \to 0$, we must have $K_i \to F_i$ when $v \to 0$ (i.e. $v_i \ll c$)

Let us consider Maxwell's EM theory, since it is covariant, with a known force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$.

The Lorentz force in terms of $\vec{A}$ and ϕ (i.e. A_μ):

$$\vec{F} = q\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \nabla \times \vec{A}\right)$$

$$\overset{\text{p.17}}{\Rightarrow} F_i = -q\left[\partial_i(\phi - \vec{v}\cdot\vec{A}) + \frac{dA_i}{dt}\right]$$

Since $u_\mu A_\mu = \gamma v_i A_i + \gamma i c \frac{i\phi}{c} = \gamma(\vec{v}\cdot\vec{A} - \phi)$

and $dt = \gamma\, d\tau$, we have

$$F_i = -q\left[-\frac{1}{\gamma}\partial_i(u_\mu A_\mu) + \frac{1}{\gamma}\frac{dA_i}{d\tau}\right]$$

$$= \frac{q}{\gamma}\left[\partial_i(u_\mu A_\mu) - \frac{dA_i}{d\tau}\right]$$

When $v \ll c$, $\gamma \to 1$, and we identify

$$K_i = q\left[\partial_i(u_\mu A_\mu) - \frac{dA_i}{d\tau}\right] = \gamma F_i$$

and $K_\mu = q\left[\partial_\mu(u_\nu A_\nu) - \frac{dA_\mu}{d\tau}\right]$

4-momentum: $p_\mu = m u_\mu = (m\gamma \vec{v}, ic\gamma m)$

$$\Rightarrow K_\mu = \frac{dp_\mu}{d\tau}$$

$$\Rightarrow K_i = \gamma F_i = \frac{dp_i}{dt/\gamma} \Rightarrow \vec{F} = \frac{d\vec{p}}{dt} \quad ☺$$

Energy and power:

$$u_\mu K_\mu = u_\mu \frac{d}{d\tau}(m u_\mu) = \frac{d}{d\tau}\left(\tfrac{1}{2} m u_\mu u_\mu\right) = 0$$

since $u_\mu u_\mu = -c^2$ (see p. 87)

$$\Rightarrow u_i K_i + u_4 K_4 = 0$$

$$\Rightarrow \gamma v_i \cdot \gamma F_i + ic\gamma \cdot K_4 = 0$$

$$\Rightarrow K_4 = \frac{i\gamma}{c} \vec{F}\cdot\vec{v} \qquad \Rightarrow K_\mu = \gamma\left(\vec{F}, \frac{i}{c}\vec{F}\cdot\vec{v}\right)$$

We combine $dp_4/d\tau = K_4$ with $p_4 = ic\gamma m$ and $K_4 = (i\gamma/c)\,\vec{F}\cdot\vec{v}$ and find

$$\vec{F}\cdot\vec{v} = \frac{d}{dt}(\gamma m c^2)$$

Since $\vec{F}\cdot\vec{v}$ is added power, we identify the total energy $E = \gamma m c^2$, and $p_4 = i\gamma m c = iE/c$

In summary: $\boxed{\vec{F} = d\vec{p}/dt, \quad \vec{F}\cdot\vec{v} = dE/dt, \quad p_\mu = (\vec{p}, iE/c)}$

- Rest energy $(v=0)$: $E_0 = mc^2$
- Kinetic energy: $T = E - E_0 = (\gamma - 1)mc^2$
- Low speed: $\gamma \approx 1 + \frac{1}{2}\beta^2 \Rightarrow T \approx \frac{1}{2}mv^2$
- If the particle is in an external field, the potential energy V must be added to γmc^2

Transformation of momentum and energy:

Since $p_\mu = m\, dx_\mu / d\tau$, p_μ will transform as x_μ,

$$\mathbb{p}' = \mathbb{L}\, \mathbb{p}$$

$$\Rightarrow \quad p_1' = p_1, \quad p_2' = p_2, \quad p_3' = \gamma p_3 + i\beta\gamma p_4 = \gamma\left(p_3 - \frac{v}{c^2}E\right)$$

$$p_4' = -i\beta\gamma p_3 + \gamma p_4 = -i\beta\gamma p_3 + \gamma \frac{iE}{c} \quad \text{and} \quad p_4' = \frac{iE'}{c}$$

$$\Rightarrow \quad E' = \gamma\left(E - v p_3\right)$$

The invariant $p_\mu p_\mu$ in the particle's rest frame ($\vec{p}=0$, $\gamma=1$):

$$p_\mu p_\mu = -E_0^2/c^2 = -m^2c^2$$

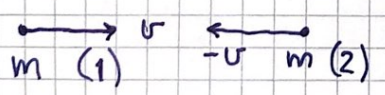
Hence, in general: $p^2 - E^2/c^2 = -m^2c^2$

$$\Rightarrow \quad \boxed{E^2 = p^2c^2 + m^2c^4}$$

Ex 1: Photons; $m = 0 \Rightarrow E = pc$. Planck: $E = h\nu$

$$\Rightarrow pc = p\lambda\nu = h\nu \Rightarrow p = h/\lambda \quad \text{(de Broglie)}$$

Ex 2: Fusion; energy conservation and mass change



$\Rightarrow$ M at rest

No external force $\Rightarrow$ conservation of $\vec{p} = \vec{p}_1 + \vec{p}_2$ $(=0)$

LT mixes E and $\vec{p}$ $\Rightarrow$ energy also conserved

$\Rightarrow$ Total 4-momentum $P_\mu = (\vec{p}, iE/c)$ conserved

Conservation of P_4: $\gamma mc^2 + \gamma mc^2 = Mc^2$

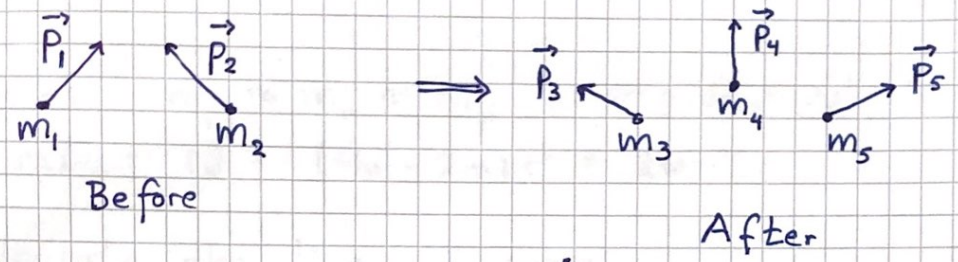
$$\Rightarrow M = 2\gamma m > 2m$$

Kin. energy $\rightarrow$ Rest energy (mass): $\Delta M = M - 2m = 2m(\gamma - 1)$

7.5 Relativistic kinematics

Relevance: Particle physics, High energy physics

Typical situation: 2 particles collide and form 2 or more new particles. Details of the interaction unknown, but conservation of momentum and energy, i.e., cons. of 4-momentum applies to "before" and "after".



Before

After

$$P_\mu = p_{1\mu} + p_{2\mu} \qquad P'_\mu = p_{3\mu} + p_{4\mu} + p_{5\mu} \quad (+\ldots)$$

For each particle: $p_\mu = (\vec{p}, iE/c)$

Convenient inertial system:

The center-of-momentum system (COM) where

$$\vec{P} = \sum_j \vec{p}_j = 0$$

In the COM system we have

$$P_\mu P_\mu = P'_\mu P'_\mu = -M^2c^2$$

which defines the equivalent mass M of the system.

Threshold energy = Minimum energy required for a reaction to take place; corresponds to all reaction products at rest in the COM system after the collision. In that case, $M = \sum_r m_r$ = sum of mass of rxn. products

The Q-value of a reaction:

$$Q = \left[\sum_r m_r - (m_1 + m_2)\right]c^2 = \text{rest energy created in the reaction}$$

Ex: Antiproton production

$$p + n \rightarrow p + n + p + \bar{p} \qquad [\text{or: } p + p \rightarrow p + p + p + \bar{p}]$$

P.A.M. Dirac, Nobel Lecture 12.12.1933: Prediction of $\bar{p}$

O. Chamberlain, E. Segrè, C. Wiegand, T. Ypsilantis, Phys. Rev. 100, 947 (1955): Observation of $\bar{p}$

[Chamberlain and Segrè, Nobel Prize 1959]

Masses: $m_p \approx m_n \approx m_{\bar{p}} \approx 938\ \text{MeV}/c^2 = m$

Q-value: $Q = (4m - 2m)c^2 = 2mc^2$

Calculate the threshold energy

(a) T_p with n at rest initially

(b) $T_p = T_n$ for head-on collision with $\vec{p}_p = -\vec{p}_n$

$\hookrightarrow$ $\boxed{\text{Assignment 11}}$

(a) We consider the covariant $P_\mu P_\mu$ in the lab system, to get T_p into the equation, and compare it with $P'_\mu P'_\mu$ after the collision in the COM system, where $M = 4m$. In other words,

$$P_\mu P_\mu = -M^2c^2 = -16m^2c^2$$

The left hand side is

$$P_\mu P_\mu = (p_{p\mu} + p_{n\mu}) \cdot (p_{p\mu} + p_{n\mu})$$

$$= p_{p\mu} p_{p\mu} + p_{n\mu} p_{n\mu} + 2 p_{p\mu} p_{n\mu}$$

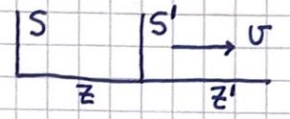
$$= -m^2c^2 - m^2c^2 + 2\left(\vec{p}_p \cdot \underset{\overset{\shortparallel}{0}}{\vec{p}_n} + \frac{iE_p}{c} \cdot \frac{iE_n}{c}\right)$$

Here, $E_n = mc^2$ and $E_p = mc^2 + T_p$

$$\Rightarrow -4m^2c^2 - 2mT_p = -16m^2c^2 \quad \Rightarrow T_p = 6mc^2 = \underline{\underline{5.57\ \text{GeV}}}$$

7.6 The electromagnetic field tensor and transformation of $\vec{E}$ and $\vec{B}$

As usual, we assume :



Definition of EM field tensor :

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

i.e. an antisymmetric tensor, $F_{\nu\mu} = -F_{\mu\nu}$

We have : $A_\mu = (\vec{A}, i\phi/c)$; $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi$; $\vec{B} = \nabla \times \vec{A}$

Hence :

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1/c \\ -B_3 & 0 & B_1 & -iE_2/c \\ B_2 & -B_1 & 0 & -iE_3/c \\ iE_1/c & iE_2/c & iE_3/c & 0 \end{pmatrix}$$

Lorentz transformation of rank 2 tensor :

$$F'_{\mu\nu} = L_{\mu\alpha} L_{\nu\beta} F_{\alpha\beta}$$

with

$$L_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}$$

Now, it is straightforward to find $\vec{E}'$ and $\vec{B}'$ expressed in terms of $\vec{E}$, $\vec{B}$ and $\vec{v}$.

Let us derive a couple of components, then write down the remaining components, and finally consider the nonrelativistic limit $v \ll c$.

$\mu = 4,\ \nu = 1$:

$$\frac{i}{c} E_1' = F_{41}' = L_{4\alpha} L_{1\beta} F_{\alpha\beta} = -i\beta\gamma B_2 + \gamma \cdot \frac{i}{c} E_1$$

$$\Rightarrow E_1' = \gamma (E_1 - v B_2)$$

Assignment 11 : $E_2' = \gamma (E_2 + v B_1)$; $E_3' = E_3$

If $v \ll c$, $\gamma = 1$, and $\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$

$i = 2,\ k = 3$:

$$B_1' = F_{23}' = L_{2\alpha} L_{3\beta} F_{\alpha\beta} = L_{33} F_{23} + L_{34} F_{24}$$

$$= \gamma B_1 + i\beta\gamma \cdot \left(-\frac{i}{c} E_2\right) = \gamma \left(B_1 + \frac{\beta}{c} E_2\right)$$

Assignment 11 : $B_2' = \gamma \left(B_2 - \frac{\beta}{c} E_1\right)$; $B_3' = B_3$

If $v \ll c$, $\gamma = 1$, and $\vec{B}' = \vec{B} - \frac{1}{c^2} (\vec{v} \times \vec{E})$

9. Canonical transformations

Related to the Hamiltonian formulation of mechanics, central in statistical mechanics and quantum mechanics.

From chapter 8:

The <u>phase space</u> is $2n$-dimensional, with n axes for the gen. coordinates q_i and n axes for the conjugate (or canonical) momenta p_i.

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = p_i(q, \dot{q}, t) \quad ; \quad i = 1, \dots, n \quad ; \quad L = L(q, \dot{q}, t)$$

$$H = p_i \dot{q}_i - L = H(p, q, t)$$

$$\left.\begin{aligned} \dot{q}_i &= \partial H / \partial p_i \\ \dot{p}_i &= -\partial H / \partial q_i \end{aligned}\right\} \text{ Hamilton's equations}$$

Ordinary coord. transformations $Q_i = Q_i(q, t)$ are called <u>point transformations</u>.

Ex: From cartesian coords. $q = \{x, y\}$ to plane polar coords. $Q = \{r, \theta\}$

Within the Hamiltonian formulation, we may transform both q_i and p_i,

$$Q_i = Q_i(q, p, t) \quad ; \quad P_i = P_i(q, p, t)$$

This transformation of phase space is called a <u>canonical transformation</u>. The new variables obey Hamilton's eqns.

$$\dot{Q}_i = \partial K / \partial P_i \quad ; \quad \dot{P}_i = -\partial K / \partial Q_i$$

with a Hamiltonian $K(Q, P, t)$.

We saw in chapter 2 that Lagrange's eqns. could be derived from Hamilton's principle,

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0$$

We will show that Hamilton's eqns. can be derived from a modified Hamilton's principle,

$$\delta \int_{t_1}^{t_2} \left[p_i \dot{q}_i - H(p, q, t) \right] dt = 0$$

The integrand $f(q, \dot{q}, p, t) = p_i \dot{q}_i - H(p, q, t)$ is independent of $\dot{p}_i$. As in ch. 2.3 (and 2.2), we arrived at Lagrange's eqns. (and Euler's eqns.) by setting

$$\Big|_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \delta q_i = 0 \qquad \text{(see p. 21 in lecture notes)}$$

since we assumed $\delta q_i(t_1) = \delta q_i(t_2) = 0$ at the outset. Now, we have automatically

$$\Big|_{t_1}^{t_2} \frac{\partial f}{\partial \dot{p}_i} \delta p_i = 0$$

since f is independent of $\dot{p}_i$. Thus, we obtain the Euler eqns. for f,

$$\frac{d}{dt} \frac{\partial f}{\partial \dot{q}_i} - \frac{\partial f}{\partial q_i} = 0 \quad ; \quad \frac{d}{dt} \frac{\partial f}{\partial \dot{p}_i} - \frac{\partial f}{\partial p_i} = 0$$

Here, $\partial f / \partial \dot{q}_i = p_i$, $\partial f / \partial q_i = -\partial H / \partial q_i$
and $\partial f / \partial \dot{p}_i = 0$, $\partial f / \partial p_i = \dot{q}_i - \partial H / \partial p_i$

$$\Rightarrow \dot{p}_i = -\partial H / \partial q_i \quad ; \quad \dot{q}_i = \partial H / \partial p_i$$

i.e., Hamilton's eqns.

The modified Hamilton's principle must also be obeyed in the new set of canonical coordinates:

$$\delta \int_{t_1}^{t_2} \left[P_i \dot{Q}_i - K(P,Q,t) \right] dt = 0$$

Now, new and old coords. and Hamiltonians are related,

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt}$$

Here, F is any ("well-behaved", i.e., with continuous 2. derivatives) function of old and/or new phase space coordinates (with zero variation in the end points, i.e., $\delta F(t_1) = \delta F(t_2) = 0$).

[To be precise, $\lambda (p_i \dot{q}_i - H) = P_i \dot{Q}_i - K + dF/dt$.
If $\lambda \neq 1$: Extended canonical transformation.
We consider only the case with $\lambda = 1$]

Particularly useful are canonical transformations where F is a function of both old and new variables.

The <u>generating function</u> F facilitates the transformation. The 4 combinations of n old and n new variables give rise to 4 types :

$$F_1(q, Q, t), \quad F_2(q, P, t), \quad F_3(p, Q, t), \quad F_4(p, P, t)$$

We examine all of them.

(1) $F = F_1(q, Q, t)$

$$\Rightarrow p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \overbrace{\frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i}^{=\frac{dF_1}{dt}}$$

Here, q_i and Q_i are considered as independent.

Then, by inspection

$$\underbrace{p_i = \frac{\partial F_1}{\partial q_i}}_{\substack{\text{n equations} \\ \text{that yield} \\ Q_i(q,p,t)}}, \quad \underbrace{P_i = -\frac{\partial F_1}{\partial Q_i}}_{\substack{\text{next these} \\ \text{n eqns. yield} \\ P_i(q,p,t)}}, \quad \underbrace{K = H + \frac{\partial F_1}{\partial t}}_{\substack{\text{finally, this eqn.} \\ \text{yields } K(Q,P,t)}}$$

(2) $F = F_2(q, P, t) - Q_i P_i$

$$\Rightarrow p_i \dot{q}_i - H = \cancel{P_i \dot{Q}_i} - K + \frac{dF_2}{dt} - \cancel{P_i \dot{Q}_i} - Q_i \dot{P}_i$$

$$= -K + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P_i}\dot{P}_i - Q_i \dot{P}_i$$

Here, q_i and P_i are considered as independent.

Then,

$$\underbrace{P_i = \frac{\partial F_2}{\partial q_i}}_{\text{yield } P_i(q,p,t)}, \quad \underbrace{Q_i = \frac{\partial F_2}{\partial P_i}}_{\text{yield } Q_i(q,p,t)}, \quad \underbrace{K = H + \frac{\partial F_2}{\partial t}}_{\text{yields } K(Q,P,t)}$$

(3) $F = F_3(p, Q, t) + q_i P_i$

$$\Rightarrow \cancel{p_i \dot{q}_i} - H = P_i \dot{Q}_i - K + \frac{dF_3}{dt} + \cancel{p_i \dot{q}_i} + q_i \dot{P}_i$$

$$\Rightarrow -H = P_i \dot{Q}_i - K + \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial p_i} \dot{p}_i + \frac{\partial F_3}{\partial Q_i} \dot{Q}_i + q_i \dot{P}_i$$

Here, p_i and Q_i are considered as independent.

$$\Rightarrow \underbrace{q_i = -\frac{\partial F_3}{\partial p_i}}_{\text{yield } Q_i(q,p,t)}, \quad \underbrace{P_i = -\frac{\partial F_3}{\partial Q_i}}_{\text{yield } P_i(q,p,t)}, \quad \underbrace{K = H + \frac{\partial F_3}{\partial t}}_{\text{yield } K(Q,P,t)}$$

(4) $F = F_4(p, P, t) + q_i p_i - Q_i P_i$

$$\Rightarrow \cancel{p_i \dot{q}_i} - H = \cancel{P_i \dot{Q}_i} - K + \frac{dF_4}{dt} + \cancel{p_i \dot{q}_i} + q_i \dot{p}_i - \cancel{P_i \dot{Q}_i} - Q_i \dot{P}_i$$

$$\Rightarrow -H = -K + \frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial p_i} \dot{p}_i + \frac{\partial F_4}{\partial P_i} \dot{P}_i + q_i \dot{p}_i - Q_i \dot{P}_i$$

Here, p_i and P_i are considered as independent.

$$\Rightarrow \underbrace{q_i = -\frac{\partial F_4}{\partial p_i}}_{\text{yield } P_i(q,p,t)}, \quad \underbrace{Q_i = \frac{\partial F_4}{\partial P_i}}_{\text{yield } Q_i(q,p,t)}, \quad \underbrace{K = H + \frac{\partial F_4}{\partial t}}_{\text{yields } K(Q,P,t)}$$

Note that both (2) and (3) involve a Legendre transformation, and (4) involves a double Legendre transformation.

Let's consider some examples.

Ex 1. $F_2 = q_i P_i$

$\Rightarrow p_i = \frac{\partial F_2}{\partial q_i} = P_i$; $Q_i = \frac{\partial F_2}{\partial P_i} = q_i$; $K = H$

$\Rightarrow F_2$ generates the identity transformation

Ex 2. $F_2 = f_i(q,t) \cdot P_i$ with arbitrary functions f_i

$\Rightarrow Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q,t)$, i.e., a point transformation, which is therefore a specific case of canonical transformations.

Ex 3. Harmonic oscillator

For a simple 1D harm. osc. $H = p^2/2m + \frac{1}{2}kq^2$

With $\omega^2 = k/m$: $H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$

We <u>know</u> that the total energy $H = E$ is conserved, i.e., independent of time.

We <u>see</u> that the transformation

$$p = f(P)\cos Q$$
$$q = [f(P)/m\omega]\sin Q$$

gives a $K = H$ independent of Q :

$$K = H = \frac{f^2(P)}{2m}\left(\cos^2 Q + \sin^2 Q\right) = \frac{f^2(P)}{2m}$$

We may eliminate $f(P)$ in the transf. above:

$f(P) = q m\omega / \sin Q \Rightarrow p = q m\omega \cot Q$

This corresponds to a generating function of the type $F_1(q, Q)$:

$p = \partial F_1 / \partial q \Rightarrow F_1 = \frac{1}{2} m\omega q^2 \cot Q$

$P = -\partial F_1 / \partial Q = m\omega q^2 / 2\sin^2 Q \Rightarrow q(Q,P) = \sqrt{\frac{2P}{m\omega}}\sin Q$

Then $f(P) = \sqrt{2m\omega P}$

and $K = f^2/2m = \omega P$

Hamilton's eqn. $\dot{P} = -\partial K/\partial Q$ shows that when Q is cyclic (K indep. of Q), the conjugate momentum P is conserved. Since $K = H = E$ = the constant total energy, we have $P = E/\omega$.

The second Hamilton eqn. is the eqn. of motion for Q:

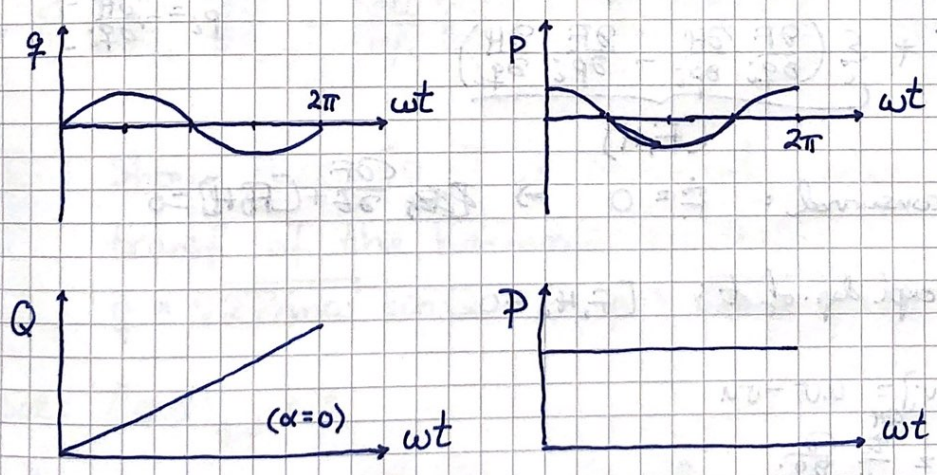
$$\dot{Q} = \partial K/\partial P = \omega \quad \Rightarrow Q(t) = \omega t + \alpha$$

where α is fixed by the initial condition $Q(0)$

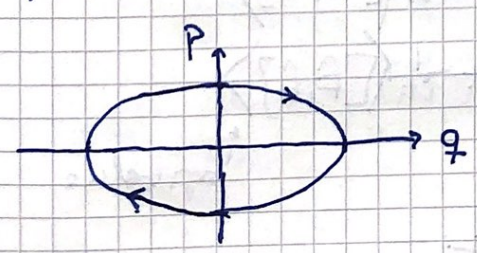
Solution in the old coordinate:

$$q = \sqrt{\frac{2P}{m\omega}}\,\sin Q = \sqrt{\frac{2E}{m\omega^2}}\,\sin(\omega t + \alpha) \quad ; \text{ not surprising}$$

Graphical illustrations:



Phase space (old coords.)



Poisson brackets:

$$[u,v] \overset{\text{def}}{=} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right) = \text{Poisson brackets}$$

of the functions u and v with respect to the set of canonical variables $(q_1 \dots q_n \,;\, p_1 \dots p_n)$.

We realize immediately:

$$[u,v] = -[v,u]$$

$$[u,c] = 0 \qquad \text{when } c = \text{const.}$$

$$[q_i, q_j] = [p_i, p_j] = 0$$

$$[q_i, p_j] = \delta_{ij}$$

Less obvious (and without proof):

$[u,v]$ is invariant under a canonical transformation

$$\Rightarrow \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right) = \sum_i \left(\frac{\partial u}{\partial Q_i} \frac{\partial v}{\partial P_i} - \frac{\partial u}{\partial P_i} \frac{\partial v}{\partial Q_i} \right)$$

Ex: Show that $[q,p]$ is invariant under the transf. of the harmonic oscillator,

$$q = \sqrt{2P/m\omega}\, \sin Q \quad ; \quad p = \sqrt{2m\omega P}\, \cos Q$$

Sol: $[q,p]_{q,p} = 1$

$$[q,p]_{Q,P} = \frac{\partial q}{\partial Q} \frac{\partial p}{\partial P} - \frac{\partial p}{\partial Q} \frac{\partial q}{\partial P}$$

$$= 2\sqrt{P} \cos Q \cos Q \cdot \tfrac{1}{2} P^{-1/2}$$

$$+ 2\sqrt{P} \sin Q \cdot \tfrac{1}{2} P^{-1/2} \sin Q$$

$$= \cos^2 Q + \sin^2 Q$$

$$= 1$$

Time dependence of an arbitrary function $F(q,p,t)$:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i \right)$$

$$= \frac{\partial F}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$= \frac{\partial F}{\partial t} + [F, H]$$

Comparison with quantum mechanics (QM):

Commutator : $[u, v]_{QM} = uv - vu$

Momentum operator : $p_i = \frac{\hbar}{i} \frac{\partial}{\partial q_i}$

$\Rightarrow [q_i, p_j]_{QM} = i\hbar \delta_{ij}$

Time dependence of QM expectation values :

$$\frac{d\langle F \rangle}{dt} = \frac{\partial \langle F \rangle}{\partial t} + \frac{1}{i\hbar} \langle [\hat{F}, \hat{H}] \rangle$$

($\hat{F}, \hat{H}$: QM operators)

A final example : Find $[L_x, L_y]$ etc. and compare with QM.

Solution :

$$\vec{L} = \vec{r} \times \vec{p}$$

$\Rightarrow L_x = y p_z - z p_y \quad ; \quad L_y = z p_x - x p_z$

$$[L_x, L_y] = \sum_{i=1}^{3} \left(\frac{\partial L_x}{\partial q_i} \frac{\partial L_y}{\partial p_i} - \frac{\partial L_x}{\partial p_i} \frac{\partial L_y}{\partial q_i} \right) \quad ; \quad q_i = x, y, z$$

$$\frac{\partial L_x}{\partial q_i} = \frac{\partial}{\partial q_i}(y p_z - z p_y) = p_z \delta_{i2} - p_y \delta_{i3}$$

$$\frac{\partial L_y}{\partial p_y} = \frac{\partial}{\partial p_i}(z p_x - x p_z) = z \delta_{i1} - x \delta_{i3}$$

$$\frac{\partial L_x}{\partial p_i} = \frac{\partial}{\partial p_i}(y p_z - z p_y) = y \delta_{i3} - z \delta_{i2}$$

$$\frac{\partial L_y}{\partial q_i} = \frac{\partial}{\partial q_i}(z p_x - x p_z) = p_x \delta_{i3} - p_z \delta_{i1}$$

$$\Rightarrow [L_x, L_y] = (p_z \delta_{i2} - p_y \delta_{i3})(z \delta_{i1} - x \delta_{i3})$$
$$- (y \delta_{i3} - z \delta_{i2})(p_x \delta_{i3} - p_z \delta_{i1})$$
$$= x p_y - y p_x = L_z$$

$$\Rightarrow [L_i, L_j] = \varepsilon_{ijk} L_k$$

From QM: $[\hat{L}_i, \hat{L}_j] = i\hbar \cdot \varepsilon_{ijk} \hat{L}_k$; $\hat{\vec{L}} = \vec{r} \times \hat{\vec{p}}$
$= \vec{r} \times \frac{\hbar}{i} \nabla$