TFY4345 Classical Mechanics. Department of Physics, NTNU.

SOLUTION ASSIGNMENT 12

Question 1

Angular momentum:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

Consider first $[p_i, L_j]$ and start with $i = j$, e.g., $i = 1$ (see p 147 in notater97.pdf):

$$\begin{aligned}
[p_x, L_x] &= [p_x, yp_z - zp_y] \\
&= [p_x, y]p_z + [p_x, p_z]y - [p_x, z]p_y - [p_x, p_y]z
\end{aligned}$$

We have

$$[p_i, x_j] = -\delta_{ij} \quad , \quad [p_i, p_j] = 0$$

and therefore

$$[p_x, L_x] = 0$$

Correspondingly:

$$[p_y, L_y] = [p_z, L_z] = 0$$

Next, consider $[p_i, L_j]$ with $i \neq j$, e.g., $i = 1$ og $j = 2$:

$$\begin{aligned}
[p_x, L_y] &= [p_x, zp_x - xp_z] \\
&= [p_x, z]p_x + [p_x, p_x]z - [p_x, x]p_z - [p_x, p_z]x \\
&= 0 + 0 - (-1)p_z - 0 \\
&= p_z
\end{aligned}$$

Cyclic change of x, y, z then gives

$$[p_y, L_z] = p_x \quad , \quad [p_z, L_y] = p_z$$

Interchanging indices only gives a change of sign, e.g.,

$$[p_z, L_y] = -p_x$$

In total,

$$[p_i, L_j] = \varepsilon_{ijk} p_k$$

With $L_j = \varepsilon_{jkl} x_k p_l$ this is obtained more directly:

$$\begin{aligned}
[p_i, L_j] &= [p_i, \varepsilon_{jkl} x_k p_l] \\
&= [p_i, x_k]\varepsilon_{jkl} p_l \\
&= -\delta_{ik}\varepsilon_{jkl} p_l \\
&= -\varepsilon_{jil} p_l \\
&= \varepsilon_{ijl} p_l
\end{aligned}$$

And now I guess we may handle $[x_i, L_j]$ the same way:

$$\begin{aligned}
[x_i, L_j] &= [x_i, \varepsilon_{jkl} x_k p_l] \\
&= [x_i, p_l]\varepsilon_{jkl} x_k \\
&= \delta_{il}\varepsilon_{jkl} x_k \\
&= \varepsilon_{jki} x_k \\
&= \varepsilon_{ijk} x_k
\end{aligned}$$

Question 2

Hand written solution in Norwegian on next page.