TFY4345 Classical Mechanics. Department of Physics, NTNU.

SOLUTION ASSIGNMENT 2

Question 1

Since there is no variation in the end points 1 and 2, we have

$$\delta \int_{t_1}^{t_2} \frac{dF}{dt}\, dt = \delta F(t_1) - \delta F(t_2) = 0,$$

and $L' = L + dF/dt$ obeys the same equations as L.

Question 2

With $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$ the equations of motion are

$$\begin{aligned} 2m\ell^2\ddot{\beta}_1 + m\ell^2\ddot{\beta}_2\cos(\beta_1-\beta_2) + m\ell^2\dot{\beta}_2^2\sin(\beta_1-\beta_2) + 2mg\ell\sin\beta_1 &= 0 \\ m\ell^2\ddot{\beta}_2 + m\ell^2\ddot{\beta}_1\cos(\beta_1-\beta_2) - m\ell^2\dot{\beta}_1^2\sin(\beta_1-\beta_2) + mg\ell\sin\beta_2 &= 0 \end{aligned}$$

Linearize the equations, i.e., $\sin x \simeq x$ and $\cos x \simeq 1$, and divide by $m\ell^2$ to obtain

$$\begin{aligned} 2\ddot{\beta}_1 + \ddot{\beta}_2 + 2\omega_0^2\beta_1 &= 0, \\ \ddot{\beta}_2 + \ddot{\beta}_1 + \omega_0^2\beta_2 &= 0. \end{aligned}$$

Next, use the ansatz $\beta_i = A_i\cos\omega t$:

$$\begin{aligned} (2\omega_0^2 - 2\omega^2)A_1 - \omega^2 A_2 &= 0, \\ -\omega^2 A_1 + (\omega_0^2 - \omega^2)A_2 &= 0. \end{aligned}$$

We have nontrivial solutions if

$$\begin{vmatrix} 2\omega_0^2 - 2\omega^2 & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0$$

or

$$\omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0$$

with solutions

$$\omega_a^2 = (2+\sqrt{2})\omega_0^2 \quad , \quad \omega_b^2 = (2-\sqrt{2})\omega_0^2$$

For each oscillation mode, the ratio A_1/A_2 is found by inserting the frequency ω_a or ω_b into one of the equations for A_1 and A_2 above. (Both equations give the same result for a given mode.) For mode a, we find $A_1/A_2 = -1/\sqrt{2}$ and for mode b, $A_1/A_2 = 1/\sqrt{2}$.

Question 3

$$\nabla\times(\nabla\times\boldsymbol{A}) = \varepsilon_{ijk}\partial_j\varepsilon_{klm}\partial_l A_m = \varepsilon_{kij}\varepsilon_{klm}\partial j\partial_l A_m = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\partial j\partial_l A_m = \partial_i(\partial_j A_j)-(\partial_j\partial_j)A_i = \nabla(\nabla\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A}$$

$$\boldsymbol{V}\times(\nabla\times\boldsymbol{V}) = \varepsilon_{ijk}V_j\varepsilon_{klm}\partial_l V_m = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})V_j\partial_l V_m = V_j\partial_i V_j - V_j\partial_j V_i = \frac{1}{2}\partial_i V_j V_j - (V_j\partial_j)V_i = \frac{1}{2}\nabla V^2 - (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}$$

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \partial_i\varepsilon_{ijk}A_jB_k = \varepsilon_{ijk}\partial_i(A_jB_k) = B_k\varepsilon_{kij}\partial_i A_j - A_j\varepsilon_{jik}\partial_i B_k = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$$

Question 4

A small displacement in the plane is $ds = \sqrt{dx^2 + dy^2}$. The total length of an arbitrary curve from point 1 to point 2 is therefore

$$I = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

The shortest path is given by the solution of Euler's equation with

$$f = \sqrt{1 + (y')^2}.$$

We have

$$\frac{\partial f}{\partial y} = 0 \quad , \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}.$$

Euler's equation is then

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = 0$$

or

$$\frac{y'}{\sqrt{1 + (y')^2}} = a$$

where a is some constant. In other words,

$$\frac{dy}{dx} = \frac{a}{\sqrt{1 - a^2}} = b,$$

and $y(x) = bx + c$, a straight line.

In polar coordinates, $ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + (r d\phi)^2} = dr\sqrt{1 + r^2(\phi')^2}$. Here, we used $x = r\cos\phi$ and $y = r\sin\phi$, and $\phi' = d\phi/dr$. The total length of an arbitrary curve from point 1 to point 2 is therefore

$$I = \int_1^2 ds = \int_{r_1}^{r_2} \sqrt{1 + r^2(\phi')^2} dr.$$

Euler's equation is now

$$\frac{d}{dr}\left(\frac{r^2\phi'}{\sqrt{1 + (r\phi')^2}}\right) = 0$$

or

$$\frac{r^2\phi'}{\sqrt{1 + (r\phi')^2}} = a.$$

We solve this equation with respect to ϕ' and find

$$\frac{d\phi}{dr} = \frac{a}{r\sqrt{r^2 - a^2}}$$

i.e.

$$\phi(r) = \int \frac{a}{r\sqrt{r^2 - a^2}}\, dr = \arctan\frac{\sqrt{r^2 - a^2}}{a} + b.$$

Here, we cheated and looked up the integral in a suitable place, e.g., wolframalpha. This is a straight line.