TFY4345 Classical Mechanics. Department of Physics, NTNU.

SOLUTION ASSIGNMENT 6

Question 1

The total energy is (see lecture notes)

$$E = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2\right) + V(r).$$

In a central potential, $mr^2\dot{\theta} = \ell$ is a conserved quantity, so we get

$$E = \frac{1}{2}m\dot{r}^2 + \left(\frac{\ell^2}{2mr^2} + V(r)\right) = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r).$$

This is an effective 1D problem, with an effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2mr^2}$$

For the particle to reach the center, it must have sufficiently high energy to overcome the potential barrier, i.e. $E > V_{\text{eff}}(r \to 0)$. This can be written as

$$Er^2 > r^2V(r) + \frac{\ell^2}{2m}, \quad r \to 0.$$

The l.h.s. goes to zero, so that the condition becomes

$$(r^2V(r))_{r\to 0} < -\frac{\ell^2}{2m}.$$

This can be fulfilled with the potential $-k/r^2$, with $k > \ell^2/2m$, or with $V(r) = -A/r^n$, with $n > 2$ and A a positive constant.

Question 2

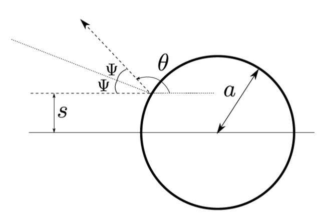

Figure 1: Hard sphere scattering, geometry.

a) The scattering angle θ satisfies $2\Psi + \theta = \pi$. From the figure, we see that the impact parameter is given by $s = a\sin(\pi/2 - \theta/2) = a\cos(\theta/2)$, so that

$$\left|\frac{ds}{d\theta}\right| = \frac{a}{2}\sin\left(\frac{\theta}{2}\right)$$

Using the formula for the differential cross section (see lecture notes) we get

$$\sigma(\theta) = \frac{s}{\sin(\theta)} \left| \frac{ds}{d\theta} \right| = \frac{a^2}{2} \frac{\cos(\theta/2)\sin(\theta/2)}{\sin(\theta)} = \frac{a^2}{4}.$$

b) The total cross section is therefore

$$\sigma = 2\pi \int_0^\pi \sigma(\theta)\sin(\theta) d\theta = \pi a^2.$$

This is physically sensible, since it is the actual cross-sectional area of the sphere.

Question 3

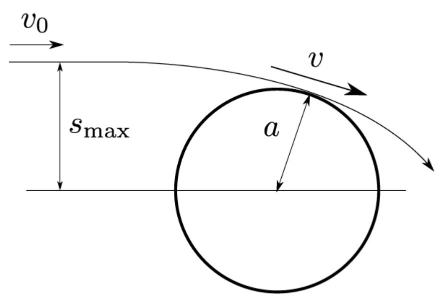

Figure 2: Gracing an attractive hard sphere.

The impact parameter s_{max} will send the particle just gracing the surface at $r = a$. Due to conservation of energy, we have

$$E = \frac{1}{2} m v_0^2 = \frac{1}{2} m v^2 - \frac{k}{a}.$$

Furthermore, conservation of angular momentum means that ℓ infinitely far away is the same as when the particle touches the surface, so

$$\ell = m v_0 s_{\text{max}} = m v a.$$

Combining these two equations, we have

$$s_{\text{max}} = \frac{v}{v_0} a = a \sqrt{1 + \frac{2k}{m a v_0^2}}.$$

All particles with impact parameter $s < s_{\text{max}}$ will hit the surface, so that $\sigma_{\text{eff}} = \pi s_{\text{max}}^2$.

Question 4

a) From the lecture notes we have

$$p = \frac{\ell^2}{mk} \qquad \varepsilon^2 = 1 + \frac{2E\ell^2}{mk^2}.$$

Eliminating ℓ gives us

$$E = -\frac{k}{2p}\left(1 - \varepsilon^2\right).$$

The total energy is constant. This means that the average total energy is also constant:

$$\langle T \rangle + \langle V \rangle = \langle E \rangle = E.$$

The virial theorem for a gravitational potential gives

$$\langle T\rangle = -\frac{1}{2}\langle V\rangle.$$

Combining this gives

$$\begin{aligned}
\langle T\rangle &= \frac{k}{2p}\left(1-\varepsilon^2\right) \\
\langle V\rangle &= -\frac{k}{p}\left(1-\varepsilon^2\right)
\end{aligned}$$

b) The solution to the Kepler problem in polar coordinates (see lecture notes) is

$$r = \frac{p}{1+\varepsilon\cos(\theta)}.$$

The average potential energy over one period is

$$\langle V\rangle = \frac{1}{t_p}\int_0^{t_p} dt\, V = -\frac{1}{t_p}\int_0^{t_p} dt\,\frac{k}{r}.$$

Combining these equations gives

$$\begin{aligned}
\langle V\rangle &= -\frac{1}{t_p}\int_0^{t_p} dt\,\frac{k}{p}\left(1+\varepsilon\cos(\theta)\right) = -\frac{k}{pt_p}\left(\int_0^{t_p} dt + \varepsilon\int_0^{t_p} dt\,\cos(\theta)\right) \\
&= \frac{k}{p}\left(1+\varepsilon\langle\cos(\theta)\rangle\right).
\end{aligned}$$

We can find the last integral by using $\ell = mr^2\dot{\theta}$ and a change of variable

$$\begin{aligned}
\langle\cos(\theta)\rangle &= \frac{1}{t_p}\int_0^{t_p} dt\,\cos(\theta) = \frac{1}{t_p}\int_0^{2\pi} d\theta\,\frac{1}{\dot{\theta}}\cos(\theta) = \frac{m}{\ell t_p}\int_0^{2\pi} d\theta\, r(\theta)^2\cos(\theta) \\
&= \frac{mp^2}{\ell t_p}\int_0^{2\pi} d\theta\,\frac{\cos(\theta)}{(1+\varepsilon\cos(\theta))^2}.
\end{aligned}$$

Using hint 2 and 3 we get

$$\begin{aligned}
\langle\cos(\theta)\rangle &= \frac{mp^2}{\ell t_p}\int_0^{2\pi} d\theta\,\frac{\cos(\theta)}{(1+\varepsilon\cos(\theta))^2} = -\frac{mp^2}{\ell t_p}\frac{d}{d\varepsilon}\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos(\theta)} \\
&= -\frac{mp^2}{\ell t_p}\frac{d}{d\varepsilon}\frac{2\pi}{\sqrt{1-\varepsilon^2}} = -\frac{2\pi m}{\ell t_p}\frac{p^2\varepsilon}{(1-\varepsilon^2)^{3/2}}.
\end{aligned}$$

Then, using

$$t_p = \frac{2\pi m}{\ell^2}\frac{p^2}{(1-\varepsilon^2)^{3/2}},$$

we get

$$\langle\cos(\theta)\rangle = -\varepsilon,$$

so

$$\langle V\rangle = \frac{k}{p}(1-\varepsilon^2).$$

c) Integrating the kinetic energy by parts, with $\dot{\boldsymbol{r}}^2 = u'v$, gives

$$\langle T\rangle = \frac{m}{2t_p}\int_0^{t_p} dt\left(\frac{d\boldsymbol{r}}{dt}\right)^2 = \frac{m}{2t_p}\left(\boldsymbol{r}\cdot\frac{d\boldsymbol{r}}{dt}\bigg|_0^{t_p} - \int_0^{t_p} dt\,\boldsymbol{r}\cdot\frac{d^2\boldsymbol{r}}{dt^2}\right)$$

Here, the first term in the parenthesis is zero, since $\boldsymbol{r}(0) = \boldsymbol{r}(t_p)$, and $\dot{\boldsymbol{r}}(0) = \dot{\boldsymbol{r}}(t_p)$. Hence,

$$\langle T\rangle = -\frac{1}{2t_p}\int_0^{t_p} dt\,\boldsymbol{r}\cdot\left(-\frac{k}{r^3}\boldsymbol{r}\right) = \frac{1}{2t_p}\int_0^{t_p} dt\,\frac{k}{r} = -\frac{1}{2}\langle V\rangle = \frac{k}{2p}(1-\varepsilon^2).$$

This agrees with the result from the virial theorem.