TFY4345 Classical Mechanics. Department of Physics, NTNU.

SOLUTION ASSIGNMENT 8

Question 1

The position of m_1 is $x_1 = x$. Kinetic energy for m_1:

$$T_1 = \frac{1}{2} m_1 \dot{x}^2$$

Kinetic energy for m_2:

$$T_2 = \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2 \right)$$

With positive θ in the figure (see assignment):

$$x_2 = x + \ell \sin\theta \quad \text{and} \quad y_2 = \ell \cos\theta$$

which yields

$$\dot{x}_2 = \dot{x} + \ell\dot{\theta}\cos\theta \quad \text{and} \quad \dot{y}_2 = -\ell\dot{\theta}\sin\theta$$

Potensial energy (only for m_2 with m_1 where $V = 0$):

$$V = -m_2 g \ell \cos\theta$$

Lagrangian:

$$\begin{aligned} L &= L(\dot{x}, \theta, \dot{\theta}) = T(\dot{x}, \theta, \dot{\theta}) - V(\theta) \\ &= \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_2}{2}\left(2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2 \right) + m_2 g \ell \cos\theta \end{aligned}$$

Here, L is independent of x (i.e., x is a cyclic coordinate), hence the canonical momentum $p_x = \partial L / \partial \dot{x}$ is constant:

$$p_x = (m_1 + m_2)\dot{x} + m_2 \ell \dot{\theta} \cos\theta = \text{ const}$$

We assume no horizontal movement of the mass center, in other words $p_x = 0$, and we may integrate the expression for p_x. This yields

$$(m_1 + m_2)x + m_2 \ell \sin\theta = \text{ const}$$

The total energy is:

$$E = T + V = \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_2}{2}\left(2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2 \right) - m_2 g \ell \cos\theta$$

We eliminate $\dot{x}$ using $p_x = 0$:

$$\dot{x} = -\frac{m_2}{m_1 + m_2}\,\ell\dot{\theta}\cos\theta$$

Hence,

$$E = \frac{m_2 \ell \dot{\theta}^2}{2}\left(1 - \frac{m_2}{m_1 + m_2}\,\cos^2\theta \right) - m_2 g \ell \cos\theta$$

We solve this equation for $\dot{\theta}$:

$$\dot{\theta} = \frac{1}{\ell}\sqrt{\frac{2\left(E + m_2 g \ell \cos\theta\right)}{m_2\left(1 - \frac{m_2}{m_1+m_2}\,\cos^2\theta\right)}}$$

Then integration yields

$$t = \ell \sqrt{\frac{m_2}{2(m_1+m_2)}} \int \sqrt{\frac{m_1 + m_2 \sin^2\theta}{E + m_2 g \ell \cos\theta}}\, d\theta$$

Here, we have rewritten somewhat:

$$1 - \frac{m_2}{m_1+m_2} \cos^2\theta = \frac{m_1 + m_2 \sin^2\theta}{m_1+m_2}$$

The system oscillates like a physical pendulum around its mass center.

Question 2

This question may be solved as in the compendium, or by inspection of a good figure. Let us use the latter method:

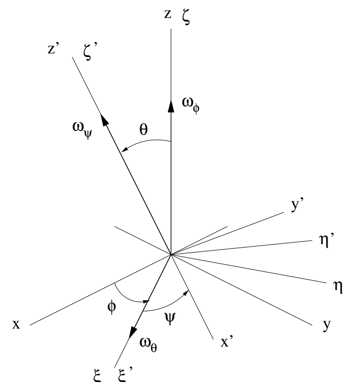

Figure 1: Euler angles.

The rotation corresponding to $\boldsymbol{\omega}$, i.e., rotation around an arbitrary axis, can be expressed as 3 successive rotations with angular velocities $\omega_\phi = \dot\phi$, $\omega_\theta = \dot\theta$ and $\omega_\psi = \dot\psi$. Here, ω_ϕ describes rotation around the z axis, i.e.,

$$\boldsymbol{\omega}_\phi = \dot\phi\, \hat z$$

Next, ω_θ describes rotaion around the ξ axis, i.e.,

$$\boldsymbol{\omega}_\theta = \dot\theta\, \hat\xi$$

Finally, ω_ψ describes rotation around the z' axis, i.e.,

$$\boldsymbol{\omega}_\psi = \dot\psi\, \hat{z}'$$

The unit vector $\hat\xi$ lies in the xy plane and may be decomposed:

$$\hat\xi = \cos\phi\, \hat x + \sin\phi\, \hat y$$

The unit vector $\hat{z}'$ has a component with length $\cos\theta$ along the z axis. The projection of $\hat{z}'$ on the xy plane has length $\sin\theta$ and points in a direction with positive x and negative y if we have rotated a positive angle ϕ as in the figure. Hence, $\hat{z}'$ has a component $\sin\theta \cdot \sin\phi$ in the x direction and a component $-\sin\theta \cdot \cos\phi$ in the y direction. In total:

$$\hat{z}' = \cos\theta\, \hat z + \sin\theta \cdot \sin\phi\, \hat x - \sin\theta \cdot \cos\phi\, \hat y$$

The total component of $\boldsymbol{\omega}$ along the x axis is therefore

$$\omega_x = \dot{\theta} \cdot \cos\phi + \dot{\psi} \cdot \sin\theta \cdot \sin\phi,$$

the total component of $\boldsymbol{\omega}$ along the y axis is

$$\omega_y = \dot{\theta} \cdot \sin\phi - \dot{\psi} \cdot \sin\theta \cdot \cos\phi,$$

and the total component of $\boldsymbol{\omega}$ along the z axis is

$$\omega_z = \dot{\phi} + \dot{\psi} \cdot \cos\theta.$$

Question 3

Viewed from a reference frame which does not rotate with the carousel, there are two forces acting on the bag: The force $\boldsymbol{F}$ applied by you on the bag (in order to keep it stationary in your lap) and the gravitational force $-mg\hat{z}$. (We may ignore effects caused by the rotation of earth, since its angular velocity is much smaller than the angular velocity of the carousel.) Then, Newtons's second law yields

$$\boldsymbol{F}_{\text{tot}} = m\boldsymbol{a} = \boldsymbol{F} - mg\,\hat{z}$$

In the lectures, we derived the relation (except for the term caused by the nonzero angular acceleration)

$$\boldsymbol{a} = \boldsymbol{a}_r + 2\boldsymbol{\omega} \times \boldsymbol{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}$$

Here, $\boldsymbol{v}_r$ and $\boldsymbol{a}_r$ are the velocity and acceleration of the bag, measured in the rotating reference frame, fixed in the carousel. You keep the bag at rest, so both vectors are zero. With positive x axis directed from the center of the carousel towards you, we have

$$\begin{aligned} \boldsymbol{\omega}(t) &= \boldsymbol{\alpha} t = \alpha t\,\hat{z} \\ \dot{\boldsymbol{\omega}} &= \boldsymbol{\alpha} = \alpha\,\hat{z} \\ \boldsymbol{r} &= r\,\hat{x} \end{aligned}$$

with numerical values $\alpha = 0.2$, $t = 10$ and $r = 5$ (SI units) at the actual instant of time. Here, the Coriolis term vanishes, since $\boldsymbol{v}_r = 0$. The centrifugal term is

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = \alpha t\,\hat{z} \times (\alpha t\,\hat{z} \times r\,\hat{x}) = \alpha^2 t^2 r\;\hat{z} \times \hat{y} = -\alpha^2 t^2 r\,\hat{x}$$

The so called Euler term is

$$\dot{\boldsymbol{\omega}} \times \boldsymbol{r} = \alpha\,\hat{z} \times r\,\hat{x} = \alpha r\,\hat{y}$$

With numbers inserted, we find $\boldsymbol{F}$ with components $F_x = -160$ N, $F_y = 8$ N and $F_z = 80$ N (with $g = 10$).

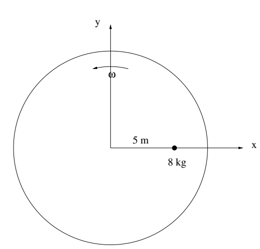

Figure 2: Carousel reference frame.