TFY4345 Classical Mechanics. Department of Physics, NTNU.

ASSIGNMENT 3 (Compulsory)

Question 1

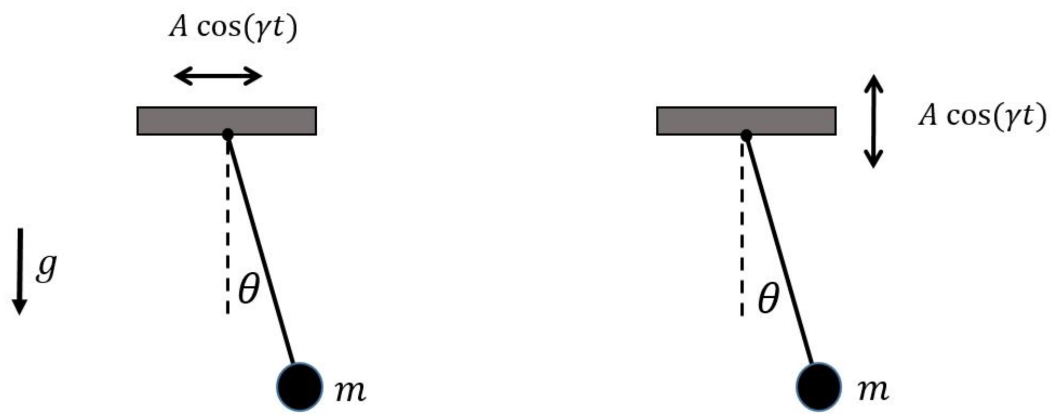

Figure 1: Pendulums attached to an oscillating point. Left: the support is oscillating horizontally. Right: the support is oscillating vertically.

(a) Consider a planar pendulum where the point of support is oscillating horizontally with a displacement $A\cos\gamma t$. Here, A is the amplitude of oscillation and γ is the frequency. Write down the Lagrangian for this system. Derive the equation of motion for θ. Consider the small angle approximation ($\theta \ll 1$) and show that the equation of motion becomes a driven harmonic oscillator. Solve this equation. What is the resonance frequency?

(b) Consider a planar pendulum where the point of support is oscillating vertically, again with a displacement $A\cos\gamma t$. Write down the Lagrangian for this system. Derive the equation of motion for θ. Show that the equation of motion has the same form as for an ordinary pendulum, but with an "effective gravitational field" having an oscillating part. (You are *not* asked to solve this equation.)

Question 2

This part of the assignment will explore the dynamics of the pendulums on oscillating supports by direct numerical integration of the equations of motion. Driven pendulums are conceptually simple mechanical systems, that nonetheless can display quite complex and sometimes non-intuitive behavior.

In Question 1, you derived the following equation of motion for the pendulum with horizontally oscillating support,

$$\ddot{\theta} = -\omega_0^2 \sin\theta + \frac{A\gamma^2}{\ell}\cos\gamma t\cos\theta. \tag{1}$$

Here, $\omega_0 = \sqrt{g/\ell}$, the oscillation frequency of small oscillations of an ordinary pendulum (with $A = 0$). For the pendulum with vertically oscillating support, you derived the equation of motion

$$\ddot{\theta} = -\omega_0^2 \sin\theta + \frac{A\gamma^2}{\ell}\cos\gamma t\sin\theta. \tag{2}$$

Numerical integration method

The equations of motion are second order differential equations in time. When doing numerical integration it is more convenient to convert a second order differential equation into two first order equations. For example, the equation with horizontally oscillating support can be expressed as

$$\begin{aligned} \dot{\theta} &= v_\theta, & (3)\\ \dot{v}_\theta &= -\omega_0^2 \sin\theta + \frac{A\gamma^2}{\ell}\cos\gamma t\cos\theta = F[\theta, t]. & (4) \end{aligned}$$

Here, $v_\theta = \dot{\theta}$ is simply the angular velocity. At time $t = 0$, we must specify initial values $\theta(0)$ and $v_\theta(0)$. We may consider the function $F[\theta, t]$ as an effective force.

There are several methods available for integrating first order equations numerically, for example Runge-Kutta. However, for Newton's equations there exists a method which is remarkably stable and simple to implement, the so-called "Verlet algorithm". Here, we will use the "velocity Verlet algorithm" to integrate the equations of motion for the pendulum. The velocity Verlet algorithm can be implemented in the following way for the pendulum:

$$\begin{aligned} v_\theta(t + \frac{\Delta t}{2}) &= v_\theta(t) + F[\theta(t), t]\,\frac{\Delta t}{2} & (5)\\ \theta(t + \Delta t) &= \theta(t) + v_\theta(t + \frac{\Delta t}{2})\,\Delta t & (6)\\ v_\theta(t + \Delta t) &= v_\theta(t + \frac{\Delta t}{2}) + F[\theta(t + \Delta t), t + \Delta t]\,\frac{\Delta t}{2} & (7) \end{aligned}$$

This is a recipe for finding $\theta(t + \Delta t)$ and $v_\theta(t + \Delta t)$ from $\theta(t)$ and $v_\theta(t)$. Notice that in the first step we calculate the velocity at the midpoint between t and $t + \Delta t$, therefore the Verlet algorithm is sometimes referred to as a midpoint method. In the second step we calculate the new "position" $\theta(t+\Delta t)$ by using the velocity at time $t + \Delta t/2$. In the third step we calculate the velocity at time $t + \Delta t$. The procedure is then repeated. Notice that the time step Δt is constant throughout the whole simulation, and it must be chosen small enough so that the resulting numerical solution is sufficiently accurate, but not too small since that would slow down the simulation. In practise, a good timestep Δt may be found by some trial and error.

Example program

The program pendulum.py is an implementation of the velocity Verlet algorithm for a simple pendulum, i.e., without an oscillating support. Both θ and v_θ are plotted as functions of time.

Part 1. Simulating a simple pendulum

Run the program pendulum.py in Python. Test the program with different values of the timestep Δt and different starting angles $\theta(0)$. You may choose $v_\theta(0) = 0$. Make sure you choose the final time such that the pendulum undergoes several oscillations, for example 10.

The equation of motion implies conservation of the total mechanical energy of the simple pendulum:

$$E = \frac{1}{2} m\ell^2\dot{\theta}^2 + mg\ell\,(1 - \cos\theta)\,. \tag{8}$$

Here, zero potential is chosen for $\theta = 0$. In a simulation the energy is not exactly conserved. The change in energy over time in a simulation can be used as a measure of the exactness of the numerical integration. For example, if the energy only changes by say 0.1% during the course of a simulation, we would normally consider this a good numerical solution. However, if the total energy changes by for example 5%, the numerical solution would be considered rather poor, and one would choose a smaller timestep.

- Calculate and plot the total energy of the simple pendulum as a function of time for different starting angles $\theta(0)$. How large can Δt (approximately) be before the energy changes by more than 0.1%?
- Calculate and plot the kinetic and potential energy. Show (Observe) that the virial theorem $\langle T\rangle = \langle V\rangle$ holds when the pendulum oscillations are small ($|\theta| \ll 1$).

Part 2. Pendulum with oscillating support

- Write a program that simulates the horizontally oscillating pendulum (Eq. 1). You may use the example program pendulum.py as a starting point.
- When the amplitude A and the driving frequency γ are both small, we expect that the driven pendulum resembles an ordinary pendulum, with some small oscillatory perturbations around the normal pendulum motion. Choose for example $A = 0.05\ell$, $\gamma = 0.2\omega_0$, and plot θ and v_θ as a function of time. Make an educated guess on a suitable timestep Δt and run one simulation. Repeat the simulation with a smaller timestep $\Delta t/2$, and check that you get the same results.
- Write a program that simulates the vertically oscillating pendulum (Eq. 2). This is very easy, you only have to replace $\cos\theta$ with $\sin\theta$ in the equation of motion.

Part 3. Resonance dynamics

The dynamics of driven pendulums may in general depend strongly both on the driving frequency γ and the driving amplitude A.

- Characterize the resonance phenomena of the horizontally driven oscillator (Eq. 1) by studying the dynamics of the pendulum as a function of the driving frequency and amplitude. The resonance can be identified by plotting the oscillation amplitude of θ and also the average total energy E as a function of γ. For small angles, you can compare your simulation results with the analytical solution valid for $\theta \ll 1$ (Question 1a).
- Characterize the resonance phenomena of the vertically driven oscillator in the same way. Notice that the resonance frequency is different from that of a horizontally driven pendulum. Hint: If $\theta(0) = 0$ and $\dot{\theta}(0) = 0$ there is no resonance, since $\ddot{\theta}(0) = 0$, as can be seen from Eq. 2. You must therefore give $\theta(0)$ a small initial value, for example $\theta(0) = 0.001$.
- Write a short text to summarize your simulation results.

Part 4. Unexpected equilibrium positions

When the driving frequency and driving amplitude are both large, the driven pendulums can exhibit very non-intuitive dynamics.

- Perform simulations with different initial angles in the range $0 \leq \theta \leq \pi$. Plot the corresponding $\theta(t)$ for different values of γ, including very high frequencies (for example up to $200\omega_0$), both for a horizontally and a vertically driven pendulum. The pendulum can in some cases perform small and rapid oscillations around a non-zero "equilibrium" angle, $\theta \neq 0$. Try to estimate for which values of γ and A these equilibrium positions occur. Hint: It can be helpful to plot the trajectories of the mass in each case, i.e., (x, y) with time t as a parameter.
- Write a short text to summarize your simulation results.

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