TFY4345 Classical Mechanics. Department of Physics, NTNU.

ASSIGNMENT 4

Question 1

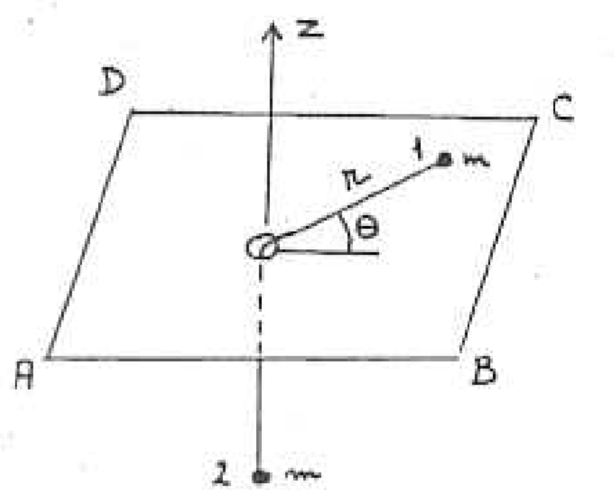

Figure 1: Two equal masses connected via a string through a hole in a table.

Two particles 1 and 2, each with mass m, are connected via a massless string with constant length s. The string is allowed to move without friction through a small hole in a horizontal table. Particle 2 can move vertically, along the z axis. Particle 1 moves horizontally without friction on the table, in the xy plane (with $z = 0$). We choose $V = 0$ in $z = 0$.

a) Show that the Lagrangian for the two particles is

$$L = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mg(r - s).$$

Here, r and θ are the plane polar coordinates of particle 1.

b) Show that the Lagrange equation for θ expresses conservation of angular momentum, $L_z = \ell =$ constant. Show that the Lagrange equation for r can be written

$$2m\ddot{r} - \frac{\ell^2}{mr^3} + mg = 0.$$

c) The equation for r allows, as a particular solution, uniform circular motion of particle 1, with $r = r_0 =$ constant. Find r_0 expressed in terms of ℓ. Is the result something you could have written down directly?

d) Assume that this uniform circular motion is slightly perturbed in the radial direction: $r \to r_0 + \rho$. In other words, $\rho(t)$ describes the small wiggle of particle 1 in its orbit around the origin. Use the Lagrange equation for r and derive the equation of motion for ρ. Linearize this equation, i.e., include only first order terms in the small (dimensionless) variable ρ/r_0. Assume initial conditions $\rho(0) = \rho_0$ and $\dot{\rho}(0) = 0$ and show that the solution is $\rho(t) = \rho_0 \cos \omega t$. Find the angular frequency ω of the wiggle movement.

Question 2

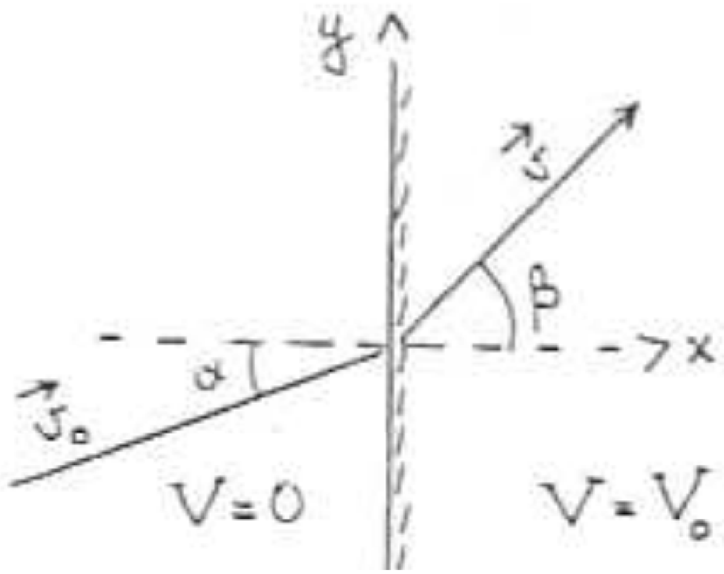

Figure 2: Potential step.

A particle with mass m comes in from the left with velocity $\boldsymbol{v}_0$ in the xy plane, in a region of space where the potential is $V = 0$. At the plane $x = 0$ there is a potential step such that $V = V_0 =$ constant for $x > 0$. The energy of the particle is $E > V_0$. Since the potential is conservative, the particle moves into the space $x > 0$ with unchanged energy but with a different constant velocity $\boldsymbol{v}$. The angle between the velocity and the x axis is α and β on the left and right side of $x = 0$, respectively.

Argue why the y component of the velocity is unchanged when the particle crosses the plane at $x = 0$.

Show that

$$\frac{\sin\alpha}{\sin\beta} = n$$

with "index of refraction"

$$n = \sqrt{1 - \frac{V_0}{E}}.$$

Question 3

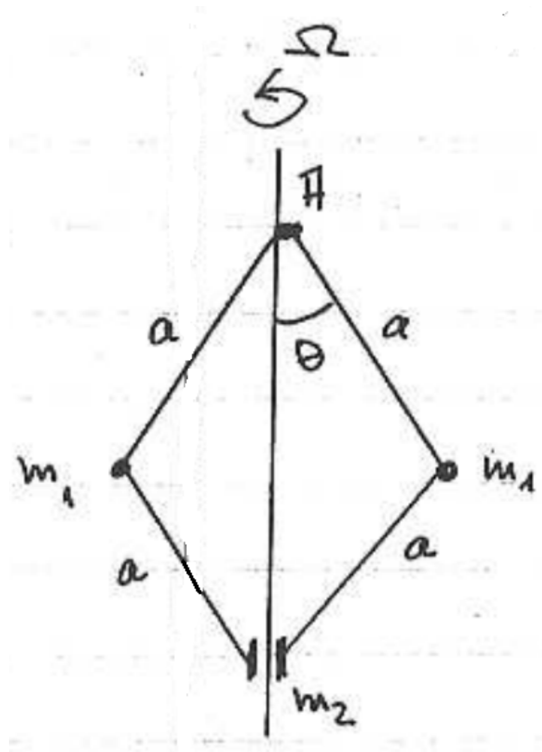

Figure 3: Rotating masses connected with massless arms.

A mass m_2 can glide vertically and without friction on a vertical rod. Two massless arms connect m_2 with masses m_1, one on each side of the vertical rod. Finally, massless arms connect the masses m_1 to the fixed point A. All four arms have equal length a. This system of masses and arms rotate around the vertical rod with constant angular velocity Ω.

a) Find the Langrangian of the system, expressed in terms of θ and $\dot{\theta}$.

b) Derive the Lagrange equation. At what constant angles θ_0 can the system be in rotational equilibrium?

c) Set $m_1 = m_2 = m$ and consider the problem as one-dimensional in the angle θ, with an effective potential

$$V'(\theta) = -ma^2(\Omega^2 \sin^2\theta + 2\omega_0^2 \cos\theta).$$

Here, we have defined $\omega_0^2 = 2g/a$. Assume $\Omega > \omega_0$ and show that V' has a minimum when $\theta = \theta_0$.